PRIME NUMBER DISTRIBUTION THEOREM

THE 3RD LANDAU PROBLEM

(LEGENDRE'S CONJECTURE)

BROCARD'S CONJECTURE

Logman Shihaliev

logman1@list.ru The 1982 Decision +994505149553

ORCID: 0000-0003-1063-4712

PROBLEM STATEMENT

Let us write the set of natural numbers in the form of a table, where each row contains consecutive numbers (note: this article does not concern the "Sieve of Eratosthenes"). We will prove that each row of this table contains at least one prime number.

table 1

1	2	$3,\ldots$	$N-1$	N
$N+1$	$N+2$	$N+3,\ldots$	$2N-1$	$2N$
$2N+1$	$2N+2$	$2N+3,\ldots$	$3N-1$	$3N$
$3N+1$	$3N+2$	$3N+3,\ldots$	$4N-1$	$4N$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$mN+1$	$mN+2$	$mN+3,\ldots$	$(m+1)N-1$	$(m+1)N$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(N-1)N+1$	$(N-1)N+2$	$(N-1)N+3,\ldots$	N^2-1	N^2
N^2+1	N^2+2	$N^2+3,\ldots$	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	$(N+1)N+3,\ldots$	$(N+2)N-1$	$(N+2)N$
$(N+2)N+1=(N+1)^2$				

ABSTRACT

This article proves that each row of the specified table contains at least one prime number. In the arbitrarily taken and first rows of the table, in parallel (simultaneously) we cross out the numbers that are multiples of the prime numbers of the set:

$L = \{2,3,5, \dots, P\}$ – the set of all primes of the first row.

At the same time, the number of deleted elements in the arbitrary and first rows of the table remains balanced (*in an arbitrary line, no more numbers are crossed out than in the first line*).

In some rows of the table, the number of numbers is (before crossing out the numbers), multiples of some prime numbers (called *critical* numbers) of the set L={2,3,5,...,P} may exceed the corresponding number in the first row by one piece. Let's call these "extra" numbers *problematic* numbers.

In any arbitrarily selected row, no more numbers are eliminated than in the first row. If necessary (theoretically), to preserve the balance of eliminated numbers, some *problematic* numbers may be left uneliminated. However, as shown in Lemma 3, *problematic* numbers ultimately disappear during the full elimination process. Moreover, the number 1 remains uneliminated in the first row. Therefore, at least one number remains uneliminated in each row of the table — a prime number.

THEOREM

For any natural numbers $N \geq 2$ and k, where $1 \leq k \leq N + 2$, there exists at least one prime number in the interval $[(k-1)N + 1, kN]$.

In other words, every full row in the above-described table contains at least one prime number.

PROOF OF THE THEOREM

It is evident that the first row of the table always contains at least one prime number.

According to Bertrand's Postulate, for any natural number $N \geq 2$, there exists a prime number in the interval $[N, 2N]$. Therefore, the second row of the table (for $N \geq 2$) also contains at least one prime number.

Now we will prove that starting from the third row and onward, every arbitrarily selected row in the table contains at least one prime number.

NOTATION

Let's use $t(m) = \left[\frac{N}{m}\right]$ and $T(m)$ to denote the number of multiples of $m \leq N$ in the first row of the table before and after the start of the strikeout process, respectively.

Similarly, using $f(m)$ and $F(m)$, we denote the number of multiples of $m \leq N$ in a randomly selected row of the table before and after the start of strikeout, respectively.

Then:

$$f(m) = t(m) + \Delta_m \Rightarrow f(m) \geq t(m) \quad \textbf{(1)}$$

Lemma 1

We prove that either $\Delta_m = 0$, or $\Delta_m = 1$.

PROOF OF LEMMA 1

Let us prove that $\Delta_m < 2$.

Let the length of the first row (i.e., number of elements) be:

$$N = (m-1) + \left(1 + \left(\left[\frac{N}{m}\right] - 1\right) \cdot m\right) + \alpha = \left[\frac{N}{m}\right] \cdot m + \alpha \qquad \textbf{(2)}$$

Where:

$$0 \le \alpha \le m - 1 \qquad \textbf{(3)}$$

Where:

$(m-1)$ is the number of elements not divisible by m at the start of the first row;

α is the number of all numbers (at the end of the first line) after the largest number, which is a multiple of m ($\alpha = 0$ in the case of $N \equiv 0 (\bmod m)$).

Assume the contrary: that there exists a row where $\Delta_m \ge 2$. Then its minimum length would be:

$$N = \left(\left(\left[\frac{N}{m}\right] + \Delta_m\right) - 1\right) \cdot m + 1 = \left(\left(\left[\frac{N}{m}\right] + 2\right) - 1\right) \cdot m + 1 = \left[\frac{N}{m}\right] \cdot m + m + 1 \qquad \textbf{(4)}$$

But using equations (3) and (4), we obtain a contradiction:

$$\left[\frac{N}{m}\right] \cdot m + \alpha = \left[\frac{N}{m}\right] \cdot m + m + 1 \Rightarrow \alpha = m + 1$$

It cannot be $\Delta_m < 0$, since the minimum values of $f(m)$ and $t(m)$ are $\left[\frac{N}{m}\right]$.

LEMMA 1 is proven.

DEFINITIONS

- A number is called a *good* number if $\Delta_m = 0$.
- A number is called a *critical* number if $\Delta_m = 1$.
- If $f(m) = t(m) + 1 = \left[\frac{N}{m}\right] + 1$, we say: the value of $f(m)$ is "increased in favor of the number" $\left[\frac{N}{m}\right] + 1$ (let's write it this way $f(m) \to \left[\frac{N}{m}\right] + 1$).
- Similarly, if $F(m) = T(m) + 1$, we say: the value of $F(m)$ is "increased in favor of the number" $T(m) + 1$ (let's write it this way $F(m) \to T(m) + 1$).
- If, in an arbitrarily selected row, $\Delta_m = 1$, then that row contains a number F (see (5), called a *problematic* number) divisible by $m\left(\left[\frac{N}{m}\right] + 1\right) = m\left[\frac{N}{m}\right] + m > N$.

$$\begin{cases} F = zm\left(\left[\frac{N}{m}\right] + 1\right) = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right) \ge z(N+1) \\ m = P_1, \quad \left[\frac{N}{P_1}\right] + 1 = P_2, \quad P_1 \to P_2 \Rightarrow P_1 \to \left[\frac{N}{P_1}\right] + 1 \Rightarrow P_1P_2 \ge N + 1 \end{cases} \qquad \textbf{(5)}$$

P_1, P_2, z – are natural numbers

PROPERTY 1

It is obvious that in the rows with indices $\{1, m+1, 2m+1, \ldots\}$, the value $\Delta_m = 0$ remains constant.

Corollary of **PROPERTY 1**

In all the specified rows of the table, m is a good number.

LEMMA 2

Let's assume that we have crossed out in the arbitrarily taken and in the first lines all the numbers that are multiples of the *good* prime $p_1 \in L$, for which the following was true:

$f(p_1) = t(p_1)$.

Let us now analyze the number of remaining (i.e., not eliminated) numbers divisible by some other prime $p_i \in L \setminus p_1$ for which initially:

$f(p_i) = \left[\frac{N}{p_i}\right] + \Delta_{p_i} = t(p_i) + \Delta_{p_i}$

And after crossing out the numbers that are multiples of $p_1 \in L$, we denote the difference $F(p_i) - T(p_i)$ as δ_{p_i}:

$F(p_i) - T(p_i) = \delta_{p_i}$ (see (8))

At the same time, it is obvious that in the first and randomly selected rows there will be no numbers that are multiples of $p_1 p_i$.

Let's prove that:

$\delta_{p_i} \leq \Delta_{p_i}$

PROOF OF LEMMA 2

According to (1), for a prime number $m = p_i$ and for a composite number $m = p_1 p_i$ we write:

$$f(p_i) = t(p_i) + \Delta_{p_i} \quad \textbf{(6)}$$

$$f(p_1 p_i) \geq t(p_1 p_i) \quad \textbf{(7)}$$

$$F(p_i) - T(p_i) = \delta_{p_i} \quad \textbf{(8)}$$

Subtract (7) from (6):

$$f(p_i) - f(p_1 p_i) \leq t(p_i) - t(p_1 p_i) + \Delta_{p_i} \quad \textbf{(9)}$$

But from definitions:

$f(p_i) - f(p_1 p_i) = F(p_i)$

$t(p_i) - t(p_1 p_i) = T(p_i)$

We substitute the last two equalities in (9) and get

$$F(p_i) \leq T(p_i) + \Delta_{p_i} \quad \textbf{(10)}$$

Compare (8) and (10), we conclude:

$\delta_{p_i} \leq \Delta_{p_i}$ **(11)**

Lemma 2 is proven.

Corollary 1 of **Lemma 2**

Good numbers do not become *critical* during the process of crossing out.

Corollary 2 of **Lemma 2**

Suppose that in an arbitrary row $\Delta_m = 1$ (that is $m \to \left[\frac{N}{m}\right] + 1$). Moreover, if $\left[\frac{N}{m}\right] + 1$ (or one of its multipliers) is a *good* number, then after crossing out the numbers that are multiples of the *good* $\left[\frac{N}{m}\right] + 1$ (or its good divisor), the number m also becomes *good*.

For example, for $N = 13$ in the third row of the table (*table 2*) $\Delta_3 = 1$. In other words, in the first row of such a table, four numbers $3, 6, 9, 12$ are multiples of 3, and in the third row there are five such numbers $27, 30, 33, 36, 39$. That is, $3 \to \left[\frac{13}{3}\right] + 1 = 5$. The number 5 in this line is a *good* number, that is, $\Delta_5 = 0$. In the third line, we cross out two numbers $(30, 35)$ that are multiples of the *good* number 5. In parallel, and in the first line, we cross out two numbers $(5, 10)$ that are multiples of the *good* number 5. In the new state of the third row of *table 2*, the number of numbers $(27, 33, 36, 39)$ that are multiples of the number 3 has become the same as in the first row $(3, 6, 9, 12)$ – four. That is, in the beginning there was $f(3) = \left[\frac{13}{3}\right] + 1 = t(3) + 1 = 4 + 1 = 5$. And after crossing out the numbers that are multiples of 5, for the number 3 it turned out $\delta_3 = 0 \Rightarrow F(3) = T(3) + \delta_3 = T(3) + 0 = 4$.

table 2

1	2	3	4	~~5~~	6	7	8	9	~~10~~	11	12	13
14	15	16	17	18	19	20	21	22	23	24	25	26
27	28	29	~~30~~	31	32	33	34	~~35~~	36	37	38	39

Corollary 3 of **Lemma 2**

At any stage of deletion, if $\Delta_p = 0$ (or $\delta_p = 0$), then in an arbitrary row of *table 1* we will delete no more numbers of multiples of a *good* p (if any) than in the first row of the table of multiples of p. In this case, there will not be a single multiple of p left in an arbitrary line.

Lemma 3

If $\Delta_p = 1$, then a critical prime p exists within an arbitrary string, and a *problematic* number F may also be present (see (5)):

$$F = zm\left(\left[\frac{N}{m}\right] + 1\right) = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right) \geq z(N+1)$$

We aim to prove that, after removing all numbers (based on previous results) divisible by the primes in the set $L = \{2,3,5, \dots, P\}$ no *problematic* number F remains in the table.

PROOF OF LEMMA 3

Proof of the contrary. Suppose that after the elimination process, some problematic F numbers remain not crossed out in a randomly selected row. Let's make a table of all possible such problematic numbers. Here $\{P_1, P_2, P_3, P_4\}$ are the set of all possible critical numbers (*table 3*):

table 3

Auxiliary lemma 3.1	**Auxiliary lemma 3.2**	**Auxiliary lemma 3.3**	**Auxiliary lemma 3.4**
$F_1 = P_1P_2P_3P_4$	$F_2 = P_1^3$	$F_3 = P_1P_2^2$	$F_4 = P_1P_2P_3$

AUXILIARY LEMMA 3.1

If the problematic number is of the form $F_1 = P_1P_2P_3P_4$, according to (5) for $\{P_\mu, P_j, P_y, P_r\} = \{P_1, P_2, P_3, P_4\}$ run inequality:

$$P_\mu \cdot P_j \geq N+1, \quad P_y \cdot P_r \geq N+1.$$

Therefore,

$$F_1 = P_1P_2P_3P_4 \geq (N+1)^2 \qquad \textbf{(12)}$$

(12) contradicts the assumption, since the number $(N+1)^2$ is outside the table.

AUXILIARY LEMMA 3.1 is proven.

AUXILIARY LEMMA 3.2

If the problematic number is of the form $F_2 = P_1^3$, then one option is possible:

$$P_1 \to P_1$$

Therefore: $P_1 \to P_1 \Rightarrow P_1 = \left[\frac{N}{P_1}\right] + 1 \Rightarrow P_1 \cdot \left(\left[\frac{N}{P_1}\right] + 1\right) = P_1^2 \Rightarrow N + 1 \leq P_1^2 < 2N$

In the second row of the table, the number P_1^2 is the smallest multiple of P_1. Let's write $P_1^2 - P_1 < N$, and continue as follows:

$$P_1^2 - N = \gamma < P_1 \Rightarrow \gamma \leq P_1 - 1 \Rightarrow P_1\gamma \leq P_1^2 - P_1 < N \Rightarrow P_1\gamma < N$$

Therefore,

$$P_1^2 - N = \gamma \Rightarrow P_1^3 - P_1N = P_1\gamma \Rightarrow P_1^3 = P_1N + P_1\gamma \qquad \textbf{(13)}$$

(13) means (since $P_1\gamma < N$) that the number $F_2 = P_1^3 = P_1N + P_1\gamma$ is in the $(P_1 + 1)$th row of the *table 1*. According to property 1, the number P_1 is *good*, which means that the number $F_2 = P_1^3$ is not *problematic*.

AUXILIARY LEMMA 3.2 is proven.

AUXILIARY LEMMA 3.3

If the problematic number has the form $F_3 = P_1P_2^2$, then there are four possible options (*table 4)*:

table 3

OPTION A	**OPTION B**	**OPTION C**	**OPTION D**
$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_2$	$P_1 \to P_2 \to P_2$
$P_1 > P_2$	$P_1 < P_2$	$P_1 > P_2$	$P_1 < P_2$

OPTION A. Here, the number P_1P_2 is the smallest number in the second row of the table, a multiple of both P_1 and P_2. So $P_2^2 < N$. Next, $\gamma < P_2 < P_1$ we write $P_1P_2 = N + \gamma$. Multiply the latter by P_2 and we get:

$F_3 = P_1P_2^2 = P_2N + P_2\gamma$

Since $\gamma < P_2$ then $P_2\gamma < N$. This means that the number $F_3 = P_1P_2^2 = P_2N + P_2\gamma$ is in the row under the number $(P_2 + 1)$. According to property 1, the number P_2 is *good*, which means that the number $F_3 = P_1P_2^2$ is not *problematic*.

OPTION B. Here, the number P_1P_2 is the smallest number in the second row of the table, a multiple of both P_1 and P_2. Next, $\gamma < P_1 < P_2$ we write $P_1P_2 = N + \gamma$. Multiply the latter by P_2 and we get:

$F_3 = P_1P_2^2 = P_2N + P_2\gamma$

Since $\gamma < P_1$ then $P_2\gamma < N$. This means that the number $F_3 = P_1P_2^2 = P_2N + P_2\gamma$ is in the row under the number $(P_2 + 1)$. According to property 1, the number P_2 is *good*, which means that the number $F_3 = P_1P_2^2$ is not *problematic*.

OPTION C. Here, the number P_2^2 is the smallest number in the second row of the *table 1*, a multiple of P_2. Let's write $P_2^2 = N + \gamma$. Multiply the latter by P_1 and we get:

$P_1P_2^2 = P_1N + P_1\gamma$

Taking into account $\gamma < P_2 < P_1$ we obtain $P_1\gamma < N$. This means that the number $F_3 = P_1P_2^2 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_3 = P_1P_2^2$ is not *problematic*.

OPTION D. Here it turns out that the numbers P_1P_2 and P_2^2 are simultaneously the smallest numbers in the second row that are multiples of P_2. And this is not possible because of $P_1 \neq P_2$.

AUXILIARY LEMMA 3.3 is proven.

AUXILIARY LEMMA 3.4

If the problematic number has the form $F_4 = P_1P_2P_3$, then there are three possible options (*table 5)*:

table 5

OPTION D	**OPTION F**	**OPTION G**
$P_1 \to P_2 \to P_3 \to P_1$	$P_1 \to P_2 \to P_3 \to P_2$	$P_1 \to P_2 \to P_3 \to P_3$

OPTION E. If $P_1 \to P_2 \to P_3 \to P_1$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

** $P_3 \to P_1$. The number P_3P_1 in the second row is the smallest multiple of P_3. It turns out $P_1P_2 < P_3P_1$.

*** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2. It turns out $P_2P_3 < P_1P_2$.

The result is a contradiction:

$$\begin{cases} P_2P_3 < P_1P_2 \Rightarrow P_3 < P_1 \\ P_2P_3 > P_3P_1 \Rightarrow P_2 > P_1 \\ P_1P_2 < P_3P_1 \Rightarrow P_2 < P_3 \end{cases} \begin{cases} P_3 < P_1 < P_2 \Rightarrow P_3 < P_2 \\ P_2 < P_3 \end{cases}$$

$F_4 = P_1P_2P_3$ is not *problematic*.

OPTION F. If $P_1 \to P_2 \to P_3 \to P_2$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

Let's write $P_2P_3 = N + \gamma$.

** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2 (it turns out that $\gamma < P_2$). Hence, $P_2P_3 < P_1P_2$.

*** $P_3 \to P_2$. The number P_2P_3 in the second row is the smallest multiple of P_3 (it turns out that $\gamma < P_3$)..

$(P_2P_3 = N + \gamma)$ multiply by P_1 and get $P_1P_2P_3 = P_1N + P_1\gamma$. Since $\gamma < P_2,\ \gamma < P_3$, then $P_1\gamma < P_1P_2 \Rightarrow P_1\gamma < N$.

This means that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_4 = P_1P_2P_3$ is not *problematic*.

OPTION G. If $P_1 \to P_2 \to P_3 \to P_3$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2. Hence, $P_2P_3 < P_1P_2$.

We will write $P_2P_3 = N + \gamma$ and multiply by P_1. $\gamma < P_2 \Rightarrow P_1\gamma < N$. This means that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_4 = P_1P_2P_3$ is not *problematic*.

AUXILIARY LEMMA 3.3 is proven.

LEMMA 3 is proven.

THEOREM is proven.

COROLLARY 1. SOLUTION OF THE 3RD LANDAU PROBLEM (Legendre's conjecture). For any natural N between N^2 and $(N+1)^2$ there is at least one prime number**.**

It is obvious that Legendre's hypothesis is a special case of the prime number distribution theorem, and for any natural N between N^2 and $(N+1)^2$ there will be at least two primes, since there are two complete rows in the specified interval (at least one prime number in each) – *table 1*.

COROLLARY 2. BROCARD'S CONJECTURE. For any natural number n between p_n^2 and p_{n+1}^2 (where $p_n > 2$ and p_{n+1} are two consecutive primes), there are at least four primes.

For any prime number $p_n > 2$, we can write as follows:

$p_n = N - 1$ and $p_n + 2 = N + 1$.

$p_{n+1} - p_n \geq 2$

Between $p_n^2 = (N-1)^2$ and $(p_n + 2)^2 = (N+1)^2$ there are four complete lines (*table 6*), each of which has at least one prime number. We take into account that the minimum difference between consecutive (starting from 3) primes is 2, and therefore we chose $p_{n+1} = N + 1$. So, the greater the difference between consecutive primes, the more primes there are between their squares.

table 6

	…	$(N-2)N$
$(N-1)^2$	…	$(N-1)N$
$(N-1)N+1$	…	N^2
N^2+1	…	$(N+1)N$
$(N+1)N+1$	…	$(N+2)N$
$(N+1)^2$	…	