PRIME NUMBER DISTRIBUTION THEOREM

LEGENDRE'S CONJECTURE (THE 3RD LANDAU PROBLEM)

BROCARD'S CONJECTURE

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ABSTRACT. The article "Theorem on the distribution of prime numbers" examines the behavior and causes of the appearance of prime numbers. The relevance of the topic lies in the fact that the consequences of the "Theorem on the distribution of prime numbers" are solutions to such open problems as Legendre's conjecture and Brocard's conjecture.

NOTE. In this article, we are not talking about the «sieve of Eratosthenes»

KEYWORDS: prime numbers, Legendre's conjecture (the 3rd Landau problem), Brocard's conjecture.

PROBLEM STATEMENT. We will write the set of natural numbers $[1, (N+2)N]$ in the form of a table, with N consecutive numbers in each row as follows (In this article, we are not talking about the «sieve of Eratosthenes»):

Таблица

1	2	$3, \dots$	$N-1$	N
$N+1$	$N+2$	$N+3, \dots$	$2N-1$	$2N$
$2N+1$	$2N+2$	$2N+3, \dots$	$3N-1$	$3N$
$3N+1$	$3N+2$	$3N+3, \dots$	$4N-1$	$4N$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$mN+1$	$mN+2$	$mN+3, \dots$	$(m+1)N-1$	$(m+1)N$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$(N-1)N+1$	$(N-1)N+2$	$(N-1)N+3, \dots$	N^2-1	N^2
N^2+1	N^2+2	$N^2+3, \dots$	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	$(N+1)N+3, \dots$	$(N+2)N-1$	$(N+2)N$
$(N+2)N+1=(N+1)^2$				

THEOREM. For any natural numbers N and k, where $1 \leq k \leq N + 2$, there is at least one prime number in the interval $[kN + 1, (k + 1)N]$.

IN OTHER WORDS: there is at least one prime number in each row of the above table. In the first and arbitrarily taken rows (except for the second row of the table) of the table, in parallel (simultaneously) we will cross out all the numbers that are multiples of each of the primes of the set L. After crossing out, we compare the number of crossed out numbers in the first and in randomly selected rows of the table. As a result, we prove that in an arbitrarily taken row of the table we cross out no more numbers than in the first row of the table. We do not cross out the crossed-out numbers again. After all the strikeouts in the first line, the «unit» remains uncrossed. Therefore, at least one number in each row of the table will remain uncrossed. This number is a prime number because it does not have a prime divisor in the set L.

NOTE. At the end of the deletion, in the first line, in addition to «unit», some other numbers may remain uncrossed. This does not contradict the presence of a «unit» in the first line and, therefore, does not contradict the fact that there is at least one prime number in an arbitrarily taken line.

ACCORDING TO BERTRAND'S POSTULATE: for any natural $N \geq 2$, there is a prime number in the interval $[N, 2N]$. Therefore, in this paper, we do not analyze the second line of the table for the presence of primes in it (we do not prove it).

CHRONOLOGY OF THE PROOF:

– Lemma 1;

– Lemma 2;

– Lemma 3;

– Continuation of Lemma 3;

– The behavior of all possible «problem» numbers $F_i = zP_1(f(P_1) + 1)$, which do not have «polite» divisors (P_1 is the smallest «critical» number);

– Consequence 1;

– Consequence.

PART I

(designations)

$L = \{2,3,5,\dots,P_q\}$ – the set of all primes in the first row of the table.

l_1, l_2 – prime numbers, $\{l_1, l_2\} \in L$.

m – natural number, $m \le N$.

An arbitrary string – any row of the table, except the first and second rows of the table.

The order of striking out (striking out) – crossing out (simultaneously) all the numbers in the first and arbitrarily selected rows of the table, multiples of some number, or some set of numbers.

$f(m)$ – the number of numbers that are multiples of the number $m \in \mathbb{N}$ in the first row of the table before the start of strikeout.

$t(m)$ – the number of numbers that are multiples of the number m in an arbitrary row of the table before the start of strikeout.

A «polite» number – if $t(m) = f(m)$, then we denote the number m as a «polite» number. Primes such as l_1, studied in Lemma 2, are also «polite».

Set of «polite» numbers – if $t(M) = f(M)$, then the set $M \neq \emptyset$ is denoted as the set of «polite» numbers

The «critical» number – if $t(m) = f(m) + 1$, then we denote the number m as a «critical» number.

$F(M)$ – the number of uncrossed numbers that are multiples of the number $m \cap M = \emptyset$ in the first row of the table after crossing out all the numbers that are multiples of all the numbers of some set $M \neq \emptyset$.

$T(M)$ – the number of uncrossed numbers that are multiples of the number $m \cap M = \emptyset$ in an arbitrarily taken row of the table after crossing out the numbers that are multiples of all the numbers in the set $M \neq \emptyset$.

A change in favor of the theorem – if, after crossing out numbers that are multiples of the terms of some set $M \neq \emptyset$, for the number $m \cap M = \emptyset$, such a change occurs as $T(m) - F(m) \le t(m) - f(m)$, then such a change is denoted by a change "in favor of the theorem".

«Increasing» the number in favor of another number – if $t(m) = f(m) + 1$, then we'll denote it like this: the number m is «increased» in favor of $f(m) + 1 = \left[\frac{N}{m}\right] + 1$, or so: $m \to f(m) + 1$; $\quad m \to \left[\frac{N}{m}\right] + 1$. Such an «increase» can last a long time: $m \to \left[\frac{N}{m}\right] + 1 \to \left[\frac{N}{\left[\frac{N}{m}\right]+1}\right] + 1 \to \left[\frac{N}{\left[\frac{N}{\left[\frac{N}{m}\right]+1}\right]+1}\right] + 1 \to, \dots$

The «problematic» number is relative to $\boldsymbol{m}$ – if $m \to f(m) + 1$ occurred in an arbitrarily taken row of the table, then the number $F = zm(f(m) + 1) = zm\left(\left[\frac{N}{m}\right] + 1\right)$ appears in such a row, which we denote as a «problem» number relative to m. Here is $z \in \mathbb{N}$.

«Indicators of the number»: $f(m), t(m), F(m), T(m)$ – let's denote it as «exponents of the number m».

Part II

(transformation of table 1)

Let's transform Table 1 and get a different interpretation of it. For the entire set of primes $L = \{2,3,5, \dots, P_q\}$, we write down an analogue of an arbitrarily taken row of Table 1:

$$\begin{cases} t(2) = f(2) + \Delta_2 \\ t(3) = f(3) + \Delta_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ t(P_q) = f(P_q) + \Delta_{P_q} \end{cases} \tag{1}$$

Part III

(properties)

Property 1. If there is no such number in an arbitrarily taken row as $F = zl_1l_2$ (in this case, there cannot be such a number in the first row of the table either), then striking out numbers that are multiples of l_1 does not affect the «indicators of the number l_2».

Property 2. For any number m, the equality $f(m) = \left[\frac{N}{m}\right]$ is true.

Property 3. $m\left[\frac{N}{m}\right]$ is the largest number in the first row of the table, a multiple of m.

Property 4. If there is an «increase» in the number m, then this happens only in favor of the number $\left[\frac{N}{m}\right]+1$. That is, $m \rightarrow f(m)+1=\left[\frac{N}{m}\right]+1$.

Property 5. As a consequence of property 4, the number $m\left(\left[\frac{N}{m}\right]+1\right)$ is in the second row of the table, and is the smallest multiple of m in the second row. Therefore, $m\left(\left[\frac{N}{m}\right]+1\right)>N$ is executed.

Property 6. The difference between the two «problematic» numbers is greater than the number N:

$z_2 m\left(\left[\frac{N}{m}\right]+1\right)-z_1 m\left(\left[\frac{N}{m}\right]+1\right)=z_3 m\left(\left[\frac{N}{m}\right]+1\right)>N.$

Therefore, there can be at most one «problematic» number in one row of the table relative to one number (there may not be any).

Property 7.1. The fact $m \rightarrow f(m)+1=\left[\frac{N}{m}\right]+1$ necessarily leads to the number $F=zm\left(\left[\frac{N}{m}\right]+1\right)$ appearing in an arbitrarily taken string, which may turn out to be «problematic».

Property 7.2. The presence of the number $F=zm\left(\left[\frac{N}{m}\right]+1\right)$ in an arbitrarily taken string does not mean that in this string $m \rightarrow f(m)+1=\left[\frac{N}{m}\right]+1$.

Property 8. If $N \equiv 0(\mathrm{mod} m)$, then $t(m)=f(m)$. In other words, when crossing out numbers that are multiples of m, we will cross out no more numbers in an arbitrarily taken row than in the first row of the table.

Property 9. $t(md+1)=f(m)$ is always true. In other words, when crossing out numbers that are multiples of the number m in the row numbered $md+1$, we will cross out no more numbers than in the first row of the table. The first numbers of such strings are numbers in the form $mdN+1$. Here $d \in \mathbb{N}$.

PART IV

(lemmas)

LEMMA 1. For the value $t(m)=f(m)+\Delta_m$, only one of two conditions is met, or $\Delta_m=0$, or $\Delta_m=1$.

Proof of **Lemma 1**.

Obviously, $f(m) = \left[\frac{N}{m}\right]$. Let's write the first row of the table as follows:

$$\underbrace{1,2,\ldots,m-1}_{m-1},\underbrace{m,\ldots,2m,\ldots,m\cdot f(m)}_{m(f(m)-1)+1},\underbrace{\ldots,N}_{x}$$

Here $0 \le x \le m-1$.

Obviously, in an arbitrarily taken row of the table, to achieve the value $t(m) = \left[\frac{N}{m}\right]$, there may be enough consecutive natural numbers in the amount of $m(f(m)-1)+1$ pieces. The presence of the remaining numbers in the amount of $m-1+x$ pieces can lead to an «increase» in $t(m)$ by only one unit, and $t(m) = \left[\frac{N}{m}\right]+1$. Due to the condition $m-1+x \le m-1+m-1 = 2m-2$, the option $t(m) = \left[\frac{N}{m}\right]+2$ is not possible for the value $t(m)$.

Lemma 1 is proved.

As a consequence of Lemma 1, the inequality $t(m) \ge f(m)$ holds.

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Lemma 2. Suppose that for a «critical» number l_1 in an arbitrary row of the table, $l_1 \to l_2$ occurred, that is, $t(l_1) = l_2 = f(l_1)+1 = \left[\frac{N}{l_1}\right]+1$. If l_2 is a «polite» number, then after crossing out the numbers that are multiples of the «polite» number l_2, the number $F = zl_1l_2 = zl_1\left(\left[\frac{N}{l_1}\right]+1\right)$ (the «problematic» number relative to l_1) is crossed out in this line, and as a result the number $l_1 1$ becomes «polite».

Proof of **Lemma 2.** The essence of the proof is that the numbers of multiples of l_1 and/or l_2 in the first and randomly selected rows are equal.

For the first row of the table:

$f(l_1) + f(l_2) - f(l_1l_2)$.

Let's substitute

$$f(l_1) = \left[\frac{N}{m_1}\right],\ f(l_2) = \left[\frac{N}{l_2}\right],\ f(l_1l_2) = \left[\frac{N}{l_1l_2}\right] = 0 \quad \text{(according to property 5).}$$

The total sum of all the numbers that are multiples of l_1 and l_2 in the first line is as follows:

$$f(l_1) + f(l_2) - f(l_1 l_2) = \left[\frac{N}{l_1}\right] + \left[\frac{N}{l_2}\right] \qquad (2)$$

Next, for an arbitrary row of the table:

$t(l_1) + t(l_2) - t(l_1 l_2)$.

Let's substitute

$t(l_1) = \left[\frac{N}{l_1}\right] + 1,\ t(l_2) = f(l_2) = \left[\frac{N}{l_2}\right]$ и $t(l_1 l_2) = \left[\frac{N}{l_1 l_2}\right] + \Delta_{l_1 l_2}$.

$\left[\frac{N}{l_1 l_2}\right] = 0$ and, since exactly $l_1 \to l_2$, then $\Delta_{l_1 l_2} = 1$.

$$t(l_1) + t(l_2) - t(l_1 l_2) = \left(\left[\frac{N}{l_1}\right] + 1\right) + \left[\frac{N}{l_2}\right] - \left(\left[\frac{N}{l_1 l_2}\right] + \Delta_{l_1 l_2}\right)$$

$$t(l_1) + t(l_2) - t(l_1 l_2) = \left(\left[\frac{N}{l_1}\right] + 1\right) + \left[\frac{N}{l_2}\right] - (0 + 1)$$

$$t(l_1) + t(l_2) - t(l_1 l_2) = \left[\frac{N}{l_1}\right] + 1 + \left[\frac{N}{l_2}\right] - 1$$

The total sum of all the numbers that are multiples of l_1 and l_2 in an arbitrary string will be as follows:

$$t(l_1) + t(l_2) - t(l_1 l_2) = \left[\frac{N}{l_1}\right] + \left[\frac{N}{l_2}\right] \qquad (3)$$

Taking into account (2) and (3)

$f(l_1) + f(l_2) - f(l_1 l_2) = t(l_1) + t(l_2) - t(l_1 l_2)$

Lemma 2 is proved.

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The behavior of the set of primes $\boldsymbol{L = \{2, 3, 5, \dots, P_q\}}$

Let's write the set $L = \{2,3,5, \dots, P_q\}$ in a different order:

$$\{l_1, l_2, \dots, l_j, \dots, l_i\} = L \qquad (4)$$

Here $\{l_1, l_2, \dots, l_j\} = L_j$ is the set of all «polite» primes, including those primes that became «polite» under Lemma 2, and $L_j \in L$.

We transform the system (1)

$$\begin{cases} t(l_1) = f(l_1) \\ t(l_2) = f(l_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ t(l_j) = f(l_j) \\ t(l_{j+1}) = f(l_{j+1}) + \Delta_{l_{j+1}} \\ t(l_i) = f(l_i) + \Delta_{l_i} \end{cases} \quad (5)$$

Let's study the behavior of two different arbitrarily taken primes l_1 and l_w. Here l_1 is a «polite» number, l_w is any prime number, and $\{l_1, l_w\} \in L$.

We cross out the numbers that are multiples of l_1. Since l_1 is a «polite» number, no more numbers were crossed out in an arbitrary row of the table than in the first row of the table.

Lemma 3. After crossing out the numbers that are multiples of the «polite» l_1, the parameters of l_w change in favor of the theorem in an arbitrarily taken row of the table.

Proof of **Lemma 3**.

Before the start of crossing out numbers that are multiples of l_1, it was like this: $t(l_1) - f(l_1) = 0$ and $t(l_w) - f(l_w) = \Delta_{l_w} \Rightarrow t(l_w) = f(l_w) + \Delta_{l_w}$ (6)

We crossed out in the first and randomly taken lines all the numbers that are multiples of the prime number l_1. It turns out that in an arbitrary line we crossed out all the numbers that are multiples of $l_1 l_w$ (if there are such numbers). As a consequence of Lemma 1, we know that

$$t(l_1 l_w) \geq f(l_1 l_w) \quad (7)$$

From (6) we subtract (7)

$$t(l_w) - t(l_1 l_w) \leq f(l_w) + \Delta_{l_w} - f(l_1 l_w) \quad (8)$$

In (8) we will replace

$$t(l_w) - t(l_1 l_w) = T(l_w) \text{ и } f(l_w) + \Delta_{l_w} - f(l_1 l_w) = F(l_w) + \Delta_{l_w}$$

We will get

$$T(l_w) \leq F(l_w) + \Delta_{l_w} \Rightarrow T(l_w) - F(l_w) \leq \Delta_{l_w} \quad (9)$$

Taking into account (6) and (9), it turns out that after crossing out numbers that are multiples of the prime number l_1, the parameters of the prime number l_w changed in favor of the theorem. That is:

It was: $t(l_w) - f(l_w) = \Delta_{l_w}$

Become: $T(l_w) - F(l_w) \leq \Delta_{l_w}$ (10)

Lemma 3 is proved.

CONTINUATION OF LEMMA 3

In the continuation of Lemma 3, we analyze the possibility of continuing to strike out numbers that are multiples of all the «polite» primes of the set $\{l_1, l_2, \dots, l_j\} = L_j$ in favor of the theorem.

We will replace $\delta_{l_w} \leq \Delta_{l_w}$ and inequality (10) are replaced by equality, and we will get:

$T(l_w) - F(l_w) = \delta_{l_w}$

Taking into account the arbitrariness of l_w, for the entire set L we write

$$\begin{cases} T(l_2) - F(l_2) = \delta_{l_2} \leq \Delta_{l_2} \\ T(l_3) - F(l_3) = \delta_{l_3} \leq \Delta_{l_3} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ T(l_j) - F(l_j) = \delta_{l_j} \leq \Delta_{l_j} \\ T(l_{j+1}) - F(l_{j+1}) = \delta_{l_{j+1}} \leq \Delta_{l_{j+1}} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ T(l_i) - F(l_i) = \delta_{l_i} \leq \Delta_{l_{ji}} \end{cases} \quad (11)$$

Here we note separately (12)

$$\begin{cases} \delta_{l_2} \leq \Delta_{l_2} \\ \delta_{l_3} \leq \Delta_{l_3} \\ \dots\dots\dots\dots\dots \\ \delta_{l_{j+1}} \leq \Delta_{l_{j+1}} \\ \delta_{l_{j+2}} \leq \Delta_{l_{j+2}} \\ \dots\dots\dots\dots\dots \\ \delta_{l_i} \leq \Delta_{l_i} \end{cases} \quad (12)$$

For the «polite» prime number l_w (in other words, for the set of «polite» primes $\{l_1, l_2, \dots, l_j\} = L_j \in L$) $\Delta_{l_w} = 0$ is true

We take into account the latter, and write it down:

$$\begin{cases} T(l_2) \leq F(l_2) \\ T(l_3) \leq F(l_3) \\ \dots\dots\dots\dots\dots\dots \\ T(l_j) \leq F(l_j) \\ T(l_{j+1}) - F(l_{j+1}) = \delta_{l_{j+1}} \leq \Delta_{l_{j+1}} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots... \\ T(l_i) - F(l_i) = \delta_{l_i} \leq \Delta_{l_{ji}} \end{cases} \tag{13}$$

Note that after crossing out the numbers that are multiples of the «polite» l_1, there were changes in favor of the theorem in an arbitrarily taken line. This is obvious when comparing the two systems (1) and (11), also separately according to the system of inequalities (12).

In the continuation of (13), we will highlight the «polite» part:

$$\begin{cases} T(l_2) \leq F(l_2) \\ T(l_3) \leq F(l_3) \\ \dots\dots\dots\dots\dots\dots \\ T(l_j) \leq F(l_j) \end{cases} \tag{14}$$

In the arbitrarily taken and first lines (14), let's try to cross out the numbers that are multiples of the next «polite» (if any) prime number (let it be l_2) of the set $\{l_1, l_2, \dots, l_j\} = L_j$.

When we crossed out numbers that are multiples of l_1 (lemma 3), the following could be:

- In an arbitrarily taken string, there was a number that was a multiple of $l_1 l_2 l_3$.
- In the process of crossing out numbers that are multiples of l_1, the number $h_1 = z_1 l_1 l_2 l_3$ was also crossed out in an arbitrary line.
- Let's assume that by the beginning of crossing out numbers that are multiples of l_1, in the first and randomly taken rows of the table there was only one number each, multiples of $l_2 l_3$, – this is the number $h_1 = z_1 l_1 l_2 l_3$ in an arbitrary line, and the number $h_2 = z_2 l_2 l_3$ in the first line. Here $(l_1, z_2) = 1$, in other words, there are no multiples of $l_1 l_2 l_3$ in the first line.
- Let's analyze two lines from (14)

$$\begin{cases} T(l_2) \leq F(l_2) \\ T(l_3) \leq F(l_3) \\ \dots\dots\dots\dots\dots\dots. \end{cases} \tag{15}$$

- After crossing out the numbers that are multiples of l_2, the first inequality disappears in (15), and the following expression with an undefined (as yet) sign remains..

$$T(l_3) \qquad \underbrace{???}_{an\ indefinite\ sign} \qquad F(l_3) \tag{16}$$

- Obviously, after crossing out the numbers that are multiples of l_2, changes will occur in the second inequality (15): Instead of $F(l_3)$ another $F(l_3) - 1$ will appear (since, unlike an arbitrary string, one number $h_2 = z_2 l_2 l_3$ is crossed out in the first line); and $T(l_3)$ will remain unchanged (since, unlike the first line, not a single number is crossed out in an arbitrarily taken line).

$$T(l_3) \qquad \underbrace{???}_{an\ indefinite\ sign} \qquad F(l_3) - 1 \tag{17}$$

- (**!!!**) The number $h_1 = z_1 l_1 l_2 l_3$ is taken into account in three places (5): and in $t(l_1)$, and in $t(l_2)$, and in $t(l_3)$. This means that after crossing out the number $h_1 = z_1 l_1 l_2 l_3$, we had to reflect this in (13), hence, in (14) and (15). In this case, instead of (15), we should have written (18)

$$\begin{cases} T(l_2) - 1 \le F(l_2) \\ T(l_3) - 1 \le F(l_3) - 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . \end{cases} \tag{18}$$

- (**!!!**) Obviously, the presence of numbers such as $h_3 = z_3 l_a l_b \cdot \dots \cdot l_c$ in arbitrarily taken rows of the table «creates an opportunity» for the appearance of new prime numbers in the same row. Here $l_a, l_b, \dots, l_c$ are a set of prime numbers, $\{l_a, l_b, \dots, l_c\} \in L$
- Here l_1, l_2, l_3 is an arbitrarily taken triple of «polite» primes from the set L.

CONCLUSION ON LEMMA 3 (TAKING INTO ACCOUNT LEMMA 2): После вычеркивания всех чисел, кратных всем «вежливым» простым числам множества L в произвольно взятой строке таблицы вычеркиваем не больше чисел, чем в первой строке таблицы.

PART V

(analysis of all kinds of «problematic» numbers)

We will illustrate the first two lines of the table with the help of drawings.

Suppose that at the next stage of crossing out in the table we encountered the first «problem» number, one of the prime divisors of which is the «critical» number – that is, the smallest «critical» prime number. Let this smallest «critical» prime number be P_1. In other words, in an arbitrarily taken row there is a «problematic» number with the smallest (across the entire table) «critical» prime divisor P_1.

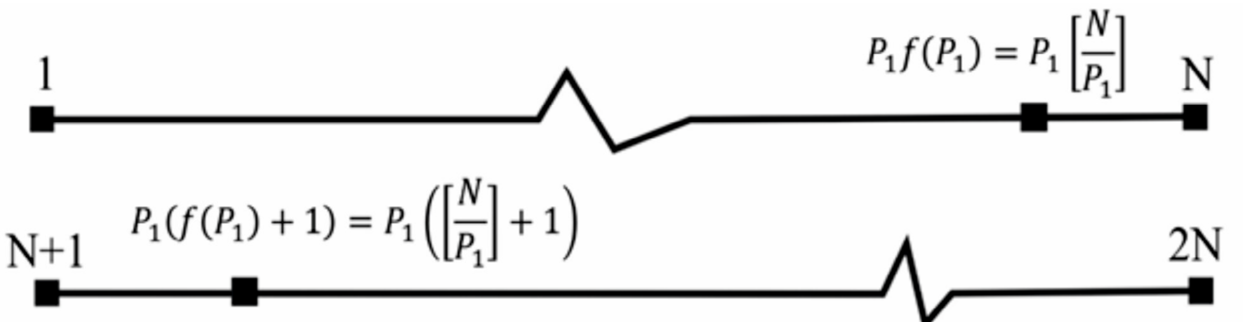

Fig 1

THE BEHAVIOR OF ALL POSSIBLE «PROBLEMATIC» NUMBERS

$$\boldsymbol{F_i = zP_1(f(P_1)+1),}$$

WHICH DO NOT HAVE «POLITE» DIVISORS AND $\boldsymbol{P_1}$

IS THE SMALLEST «CRITICAL» NUMBER IN THE ENTIRE TABLE

In the proof of Lemma 2, we studied the behavior of numbers such as $F_1 = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right]+1\right)$, where P_1 is a «critical» number, and «increased» in favor of the «polite» number P_2. As a result, P_1 becomes «polite» (Lemma 2).

Now suppose that in an arbitrarily taken string there is a «problem» number $F_i = zP_1P_2 = zP_1(f(P_1)+1)$, where the numbers P_1 and $f(P_1)+1$ are both different «critical» primes. That is, $P_1 \neq f(P_1)+1$.

Option 1. Suppose that in an arbitrary row of the table, the smallest «critical» number P_1 is «increased» in favor of the prime number $P_2 = f(P_1)+1 = \left(\left[\frac{N}{P_1}\right]+1\right)$. At the same time, the number $f(P_1)+1$ is a «critical» prime number (meaning $P_2 > P_1$), and is «increased» in favor of the prime number P_1, that is $\left(\left[\frac{N}{f(P_1)+1}\right]+1\right) = P_1$. A «problematic» number $F_2 = zP_1P_2 = zP_1(f(P_1)+1)$ appears in an arbitrarily taken row of the table.

The first two rows of the table are presented as in Figure 1. Obviously, if $z = 1$, then the number $F_2 = P_1P_2 = P_1\left(\left[\frac{N}{P_1}\right]+1\right)$ is in the second row of the table. If $z > 1$, and has a «polite» divisor (let it be v), then the «problematic» number F_2 is crossed out when we cross out numbers that are multiples of the «polite» v.

Consider the «problem» number $F_3 = zP_1P_2$, where P_1 and P_2 are both «critical» numbers, $z > 1$ and does not have a «polite» prime divisor. That is, all prime divisors of the number z are «critical». Given that P_1 is the smallest «critical» prime number, then $P_2 > P_1$. There cannot be $z = P_3P_4$, where P_3 and P_4 are «critical» primes, and P_3 «increases» in favor of P_4 (or vice versa). Figure 1 shows that the number $P_1P_2 = P_1\left(\left[\frac{N}{P_1}\right]+1\right)$ and the number $P_3P_4 = P_3\left(\left[\frac{N}{P_3}\right]+1\right)$ are both in the second row of the table. The conditions $P_1P_2 \geq N+1$ and $P_3P_4 \geq N+1$ are satisfied. It turns out $F_3 \geq (N+1)^2$. This means that the number $F_3 = P_1P_2P_3P_4$ is outside the table.

Option 2. Next, consider the options if the «increase» of the smallest «critical» number P_1 continues like this: $P_1 \to P_2 \to \cdots \to P_j \to \cdots \to P_n$. And the «problem» number will be like this: $F_4 = P_1P_2 \cdot \ldots \cdot P_j \cdot \ldots \cdot P_n$. There are no «polite» numbers among the natural divisors of the «problematic» number F_4. This means that the «critical» number P_n is also «increased». There are two possible options here: or $P_1 \to P_2,\ P_n \to P_j$, or $P_1 \to P_2,\ P_n \to P_1$. Here $j = 2, \ldots, n-1$, and $n \geq 4$.

Fig 2

Figure 2 shows that regardless of the fact that $P_1P_2 > P_nP_j$ or $P_nP_j > P_1P_2$, the conditions $P_1P_2 \geq N+1$ and $P_nP_j \geq N+1$ are met. Means, $F_4 \geq (N+1)^2$.

Option 3. Consider the variant $F_5 = P_1P_2P_3$, where either $P_1 \to P_2 \to P_3 \to P_1$, or $P_1 \to P_2 \to P_3 \to P_2$.

Figure 3 shows the following contradiction. According to property 5 in the second row, the number P_1P_2 is the smallest number that is a multiple of the number P_1 (since $P_1 \to P_2$), just as the number P_2P_3 is the smallest number in the second row that is a multiple of the number P_2. However, these two conditions are contradicted by the location of the numbers P_3P_1 (for $P_3 \to P_1$) and P_3P_2 (for $P_3 \to P_2$).

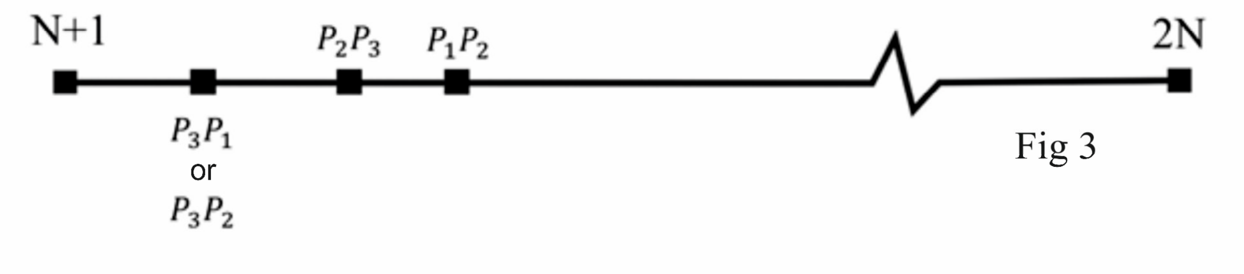

Fig 3

Option 4. Let's consider the case when there is a «problem» number $F_6 = P_1P_2^2$ in an arbitrarily taken string, where $P_1 \to P_2$ and $P_2 \to P_2$.

There are two possible options here. Either $P_2 > P_1$, or $P_1 > P_2$.

If $P_2 > P_1$, then in Figure 4 the numbers P_2P_2 and P_1P_2 will swap places, and this contradicts property 9 (for P_2). Now let's look at the option when $P_1 > P_2$.

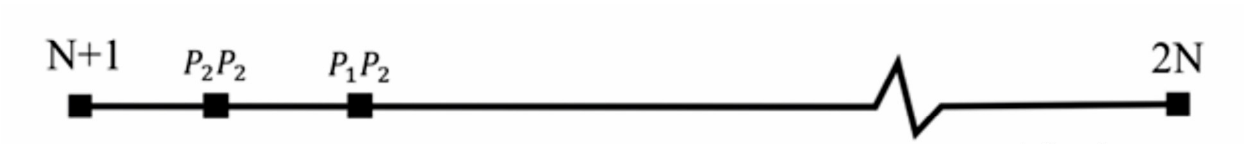

Fig 4

Let's write it down $P_2P_2 = N + \gamma \xRightarrow[\gamma < P_2 < P_1]{} P_1P_2^2 = P_1(N + \gamma) = P_1N + P_1\gamma$. Since the number P_1P_2 in the second row is the smallest multiple of P_1 (property 5), then $P_1\gamma < N$ (for comparison, we take into account $P_1(P_2 - 1) < N$). Therefore, the number $F_6 = P_1P_2^2 = P_1N + P_1\gamma$ is in the row numbered $P_1 + 1$. Given property 9, the number P_1 is «polite», and the number $F_6 = P_1P_2^2$ is not «problematic».

Option 5. Let's consider the case when there is a «problem» number $F_7 = P_1P_2^2$ in an arbitrarily taken string, where $P_1 \to P_2$ and $P_2 \to P_1$.

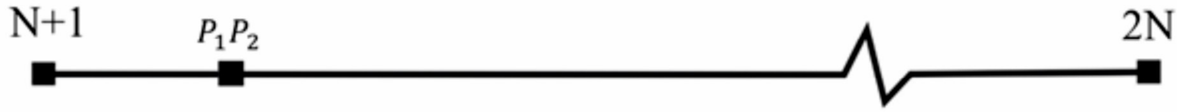

Fig 5

Let's write it down (Figure 5)

$$P_1P_2 = N + \gamma \xRightarrow[\gamma<P_1 \text{ и } \gamma<P_2]{} P_1P_2^2 = P_2(N+\gamma) = P_2N + P_2\gamma$$

Since the number P_1P_2 in the second line is the smallest number that is simultaneously a multiple of P_1 and P_2 (property 5), then $P_2\gamma < N$. Therefore, the number $F_7 = P_1N + P_1\gamma$ is in the row numbered $P_1 + 1$. Given property 9, the number P_1 is «polite», and the number $F_7 = P_1P_2^2$ is not «problematic».

Option 6. Let's consider the case when there is a «problem» number $F_8 = P_1^\alpha$ in an arbitrarily taken string, where $P_1 \to P_1$. Obviously, $\alpha = 3$. Otherwise (Figure 6),

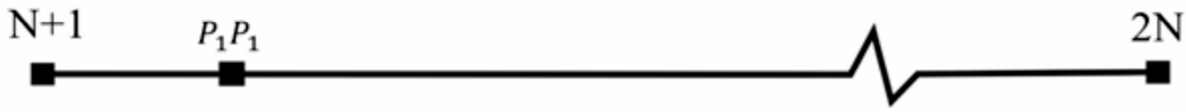

Fig 6

either $F_8 = P_1^4 \geq (N+1)^2$ (the number is outside the table), or the number $F_8 = P_1^2$ is in the second row. So $F_8 = P_1^3$. Let's write down

$$P_1P_1 = N + \gamma \xRightarrow[\gamma<P_1]{} P_1^3 = P_1(N+\gamma) = P_1N + P_1\gamma$$

Since the number P_1P_1 in the second row is the smallest multiple of P_1 (property 5), then $P_1\gamma < N$. Therefore, the number $F_8 = P_1^3 = P_1N + P_1\gamma$ is in the row numbered $P_1 + 1$. Given property 9, the number P_1 is «polite» and $F_8 = P_1^3$ is not «problematic».

The theorem on the distribution of prime numbers has been proved.

CONSEQUENCE 1. LEGENDRE'S CONJECTURE (THE 3RD LANDAU PROBLEM)**.** For any natural N between N^2 and $(N+1)^2$ there is at least one prime number.

It is obvious that Legendre's hypothesis is a special case of the theorem on the distribution of prime numbers, and for any natural N between N^2 and $(N+1)^2$ there are at least two prime numbers, since there are two complete rows in the specified interval (at least one prime number in each).

CONSEQUENCE 2. BROCARD'S CONJECTURE. For any natural number n between p_n^2 and p_{n+1}^2 (where $p_n > 2$ and p_{n+1} are two consecutive primes), there are at least four primes.

For any prime number $p_n > 2$, we can write as follows:

$p_n = N - 1$ and $p_n + 2 = N + 1$

The extreme numbers p_n^2 and p_{n+1}^2 both are composite numbers. Between $p_n^2 = (N-1)^2$ and $(p_n + 2)^2 = (N+1)^2$ there are four complete lines, each of which has at least one prime number. We take into account that the minimum difference between consecutive (starting from three) primes is 2, и поэтому выбрали $p_n = N - 1$ и $p_n + 2 = N + 1$. So, the greater the difference between consecutive primes, the more primes there are between their squares.

...	...	...	...	$(N-2)N$
$(N-1)^2$	$(N-1)^2+1$	...	$(N-1)N-1$	$(N-1)N$
$(N-1)N+1$	$(N-1)N+2$	...	N^2-1	N^2
N^2+1	N^2+2	...	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	...	$(N+2)N-1$	$(N+2)N$
$(N+1)^2$	...	...	...	...