УДК 511.333

PRIME NUMBER DISTRIBUTION THEOREM

LEGENDRE'S CONJECTURE

BROCARD'S CONJECTURE

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The article " Prime Number Distribution Theorem" examines the behavior of primes in order to determine the causes and places of their infinite occurrence. The solutions of Legendre's conjecture and Brocard's conjecture are consequences of the "Prime Number Distribution Theorem".

Keywords: prime numbers, Legendre's conjecture, Brocard's conjecture.

1. Problem statement. We will write the set of natural numbers $[1, (N+2)N]$ in the form of a table, with N consecutive numbers in each row as follows:

Table 1

1	2	3,…	$N-1$	N
$N+1$	$N+2$	$N+3, \ldots$	$2N-1$	$2N$
$2N+1$	$2N+2$	$2N+3, \ldots$	$3N-1$	$3N$
$3N+1$	$3N+2$	$3N+3, \ldots$	$4N-1$	$4N$
…	…	…	…	…
$gN+1$	$gN+2$	$gN+3, \ldots$	$(g+1)N-1$	$(g+1)N$
…	…	…	…	…
$(N-1)N+1$	$(N-1)N+2$	$(N-1)N+3, \ldots$	N^2-1	N^2
N^2+1	N^2+2	$N^2+3, \ldots$	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	$(N+1)N+3, \ldots$	$(N+2)N-1$	$(N+2)N$
$(N+2)N+1=(N+1)^2$				

We will prove a theorem on the distribution of primes. This theorem has several interpretations. Exactly:

There is at least one prime number in each complete row of Table 1.

Another version of the theorem: For any natural numbers N and L, where $1 \leq k \leq N+2$, there is at least one prime number in the interval $[kN+1, (k+1)N]$

Chronology of the proof: Lemma 1; Auxiliary theorem; The behavior of all possible "problem" numbers $F_i = zP_1(f(P_1)+1)$, which do not have "polite" divisors and P_1 is the smallest "critical" number; Consequence 1; Consequence 2.

Definitions.

N – natural number, $N \geq 2$.

z – natural number.

$L = \{2,3,5, \dots, P_{max}\}$ – the set of all primes in the first row of the table.

P_i – prime number, $P_i \in L$, $(i = 1, 2, \dots, max)$.

$L_l = \{l_1, l_2, \dots, l_{i-1}\}$ – the set of prime numbers, $L_l \in L$.

l_i – prime number, $l_i \in L$.

$f(l_i)$ – the number of numbers that are multiples of the prime number l_i in the first row of the table before striking out.

$t(l_i)$ – the number of numbers that are multiples of the prime number $l_i \in L$ in an arbitrarily taken row of the table before striking out.

The difference is denoted as follows: $t(l_i) - f(l_i) = \Delta_l$. Obviously, either $\Delta_l = 0$ or $\Delta_l = 1$.

$f(P_i)$ – the number of multiples of P_i in the first row of the table, $f(P_i) = \left[\frac{N}{P_i}\right]$.

In some rows of the table, this indicator $f(P_i) = \left[\frac{N}{P_i}\right]$ is preserved. For such strings, we denote the number P_i as a "**polite**" number. At the same time, it is obvious that in some other row of the table, the number of numbers $t(P_i)$, multiples of P_i, can be one (and only one, this is obvious) piece more $t(P_i) = \left[\frac{N}{P_i}\right] + 1$. So, in this row of the table there is a number that is a multiple of the number $P_i\left(\left[\frac{N}{P_i}\right] + 1\right)$.

For such lines, we denote the number P_i as a "**critical**" number. And we denote the number $F = zP_i\left(\left[\frac{N}{P_i}\right] + 1\right)$ as a "**problem**" number. The appearance of a "problematic" number in a row of the table is denoted as an "**increase**" of the number P_i in "**favor**" of the number $\left[\frac{N}{P_i}\right] + 1$.

"Increase" is denoted by →. For example, if P_1 is "increased" in "favor" of P_2, then $P_1 \to P_2$

or $P_1 \to \left[\frac{N}{P_1}\right] + 1$. If $P_1 \to P_2 \to P_3$, then $P_1 \to \left[\frac{N}{P_1}\right] + 1 \to \left[\frac{N}{\left[\frac{N}{P_1}\right]+1}\right] + 1$, if $P_1 \to$

$P_2 \to P_3 \to P_4$, then $P_1 \to \left[\frac{N}{P_1}\right] + 1 \to \left[\frac{N}{\left[\frac{N}{P_1}\right]+1}\right] + 1 \to \left[\frac{N}{\left[\frac{N}{\left[\frac{N}{P_1}\right]+1}\right]+1}\right] + 1 \to, \ldots$

etc.

$F(l_i)$ – is the number of non–crossed out numbers that are multiples of the prime number l_i in the first row of the table after crossing out the numbers that are multiples of all the primes of the set L_l.

$T(l_i)$ – is the number of non–crossed out numbers that are multiples of the prime number $l_i \in L$ in an arbitrarily taken row of the table after crossing out numbers that are multiples of all the primes of the set L_l.

The difference is denoted as follows: $T(l_i) - F(l_i) = \delta_l$.

$p\left[\frac{N}{p}\right]$ – is the largest number in the first row of the table, a multiple of the natural number p.

Property 1. If the "critical" number P_1 is "increased" in "favor" of the number P_2, then the number P_1P_2 is the smallest number in the second row of the table, a multiple of P_1.

2. Introduction. In the first and randomly taken (except the second) in the rows of the table, in parallel (simultaneously), we will cross out all the numbers that are multiples of the primes of the set L. After crossing out, we compare (as a result) the number of crossed out numbers (their differences) in the first and in randomly taken rows of the table (in this article, we are not talking about the "sieve of Eratosthenes"). For example, if we cross out all the numbers in the table that are multiples of N and compare the results, we will make sure that in an arbitrarily taken row we will cross out no more numbers than in the first row of the table.

Crossed out numbers are not crossed out again.

According to Bertrand's postulate: for any natural $N \geq 2$, there is a prime number in the interval $[N, 2N]$. Therefore, in this paper, we do not analyze the second line of the table for the presence of primes in it (we do not prove it).

Crossing out numbers that are multiples of a particular prime number is carried out in such a way (it turns out) that as a result, in an arbitrarily taken row of the table, we cross out no more numbers than in the first row of the table. After crossing out the numbers in the table that are multiples of each prime number of the set L, at least one number, "1", always remains uncrossed in the first row of the table (this is obvious). This means that in an arbitrarily taken string, too, at least one number remains uncrossed, which is a prime number.

Suppose that all the numbers that are multiples of each number of the set of primes $L_l = \{l_1, l_2, \dots, l_{i-1}\} \in L$ have been crossed out in the table.

$t(l_i)$ – the number of numbers that are multiples of the prime number $l_i \in L$ in an arbitrarily taken row of the table before the start of strikethrough.

$f(l_i)$ – the number of numbers that are multiples of the prime number $l_i \in L$ in the first row of the table before striking out.

The difference is denoted as follows:

$$t(l_i) - f(l_i) = \Delta_l \Rightarrow t(l_i) = f(l_i) + \Delta_l, \tag{1}$$

where $\Delta_l = 0$, or $\Delta_l = 1$.

$T(l_i)$ – is the number of non–crossed out numbers that are multiples of the prime number $l_i \in L$ in an arbitrarily taken row of the table after crossing out the numbers to all the primes of the set $L_l = \{l_1, l_2, \dots, l_{i-1}\} \in L$.

$F(l_i)$ – is the number of non–crossed out numbers that are multiples of the prime number $l_i \in L$ in the first row of the table after crossing out the numbers to the set of prime numbers $L_l = \{l_1, l_2, \dots, l_{i-1}\} \in L$.

The difference is denoted as follows:

$$T(l_i) - F(l_i) = \delta_l. \tag{2}$$

Lemma 1 (the conservation of order lemma). For the values of Δ_l and δ_l, the inequality $\delta_l \leq \Delta_l$ is satisfied.

Proof of Lemma 1. Suppose that in an arbitrarily taken string, when crossing out numbers that are multiples of a prime number (let it be l_1) from the set $L_l = \{l_1, l_2, \dots, l_{i-1}\} \in L$, numbers that are multiples of l_i were crossed out.

Consequently, numbers (one or more) multiples of $l_1 l_i$ were crossed out. We know that for a natural number $l_1 l_i$, the condition (3), (4), (5) is satisfied

$$t(l_1 l_i) \geq f(l_1 l_i) \tag{3}$$

$$t(l_i) - t(l_1 l_i) = T(l_i) \tag{4}$$

$$f(l_i) - f(l_1 l_i) = F(l_i) \tag{5}$$

From (1) we subtract (3), we also take into account (4), (5) and (2)

$$t(l_i) - t(l_1 l_i) \leq f(l_i) + \Delta_l - f(l_1 l_i) \Rightarrow T(l_i) \leq F(l_i) + \Delta_l \Rightarrow$$

$$\Rightarrow F(l_i) + \delta_l \leq F(l_i) + \Delta_l \Rightarrow \delta_l \leq \Delta_l$$

Lemma 1 is proved.

Auxiliary theorem. In each row of the table where there is no "critical" number, there is at least one prime number.

Proof.

Suppose that the number P_1 for an arbitrarily taken row of the table is "critical", and "increased" in "favor" of the "polite" number P_2, that is, $P_2 = \left[\frac{N}{P_1}\right] + 1$. It can be assumed that some "problematic" number appeared in this line

$F_1 = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right)$.

In the first row of the table there are $\left[\frac{N}{P_1}\right]$ pieces of numbers that are multiples of P_1, and $\left[\frac{N}{P_2}\right]$ pieces of numbers, multiples of P_2. There is no number in the first line that is a multiple of $P_1P_2 = P_1\left(\left[\frac{N}{P_1}\right] + 1\right)$. In total, in the first row of the table, the number of numbers that are multiples of P_1 and/or P_2 will be

$$f(P_1) + f(P_2) - f(P_1P_2) = \left[\frac{N}{P_1}\right] + \left[\frac{N}{P_2}\right] - \left[\frac{N}{P_1P_2}\right] = \left[\frac{N}{P_1}\right] + \left[\frac{N}{P_2}\right]$$

In an arbitrarily taken row of the table there are $\left[\frac{N}{P_1}\right] + 1$ pieces of numbers that are multiples of P_1, В произвольно взятой строке таблицы имеется $\left[\frac{N}{P_1}\right] + 1$ штук чисел, кратные P_1, and $\left[\frac{N}{P_2}\right]$ numbers, multiples of P_2. In total, in an arbitrarily taken row of the table, the number of numbers that are multiples of P_1 and/or P_2 will be $\quad t(P_1)+t(P_2)-t(P_1P_2)=\left(\left[\frac{N}{P_1}\right]+1\right)+\left[\frac{N}{P_2}\right]-\left(\left[\frac{N}{P_1P_2}\right]+\Delta\right)$

More than one of these "problematic" numbers $F_1=zP_2P_1$ does not fit in one line of the table, and here $\Delta=1$. When crossing out numbers that are multiples of the "polite" number P_2, the "problematic" number $F_1=zP_2P_1$ is also crossed out in an arbitrarily taken row of the table, as a result of which the number P_1 also becomes "polite".

$$\left[\frac{N}{P_1}\right]+\left[\frac{N}{P_2}\right]=\left[\frac{N}{P_1}\right]+1+\left[\frac{N}{P_2}\right]-\left(\left[\frac{N}{P_1P_2}\right]+\Delta\right)$$

Taking into account Lemma 1, it turns out that no more numbers are crossed out in an arbitrarily taken line than in the first line. And in the first line at least one number is not crossed out (this is "1"). This means that in an arbitrarily taken string, at least one number remains uncrossed, which is a prime number.

The auxiliary theorem is proved.

3. Suppose that at the next stage of crossing out in the table we encountered the first "critical" number – that is, the smallest "critical" prime number. Let it be P_1. In other words, in an arbitrarily taken string there is a "problematic" number with a natural divisor P_1.

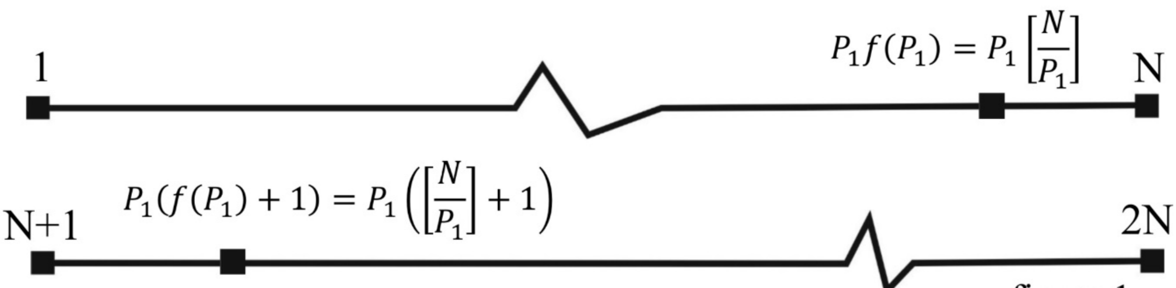

figure 1

4. The behavior of all possible "problem" numbers $F_i=zP_1(f(P_1)+1)$, which do not have "polite" divisors and P_1 is the smallest "critical" number.

In the proof of the auxiliary theorem, we studied the behavior of the number $F_1=zP_1P_2=zP_1\left(\left[\frac{N}{P_1}\right]+1\right)$, where P_1 is the only "critical" number, and "increased" in "favor" of the "polite" number P_2.

And now suppose that in an arbitrarily taken string there is a "problem" number

$F_i = zP_1P_2 = zP_1(f(P_1)+1)$, where the numbers P_1 and $f(P_1)+1$ are both different "critical" primes. That is, $P_1 \neq f(P_1)+1$.

Option 1. Suppose that in an arbitrarily taken row of the table, the smallest "critical" number P_1 is "increased" in "favor" of the prime number $P_2 = f(P_1)+1 = \left(\left[\frac{N}{P_1}\right]+1\right)$. At the same time, the number $f(P_1)+1$ is a "critical" prime number (meaning $f(P_1)+1$), and is "increased" in "favor" of the prime number P_1, that is, $\left(\left[\frac{N}{f(P_1)+1}\right]+1\right) = P_1$. A "problem" number $F_2 = zP_1P_2 = zP_1(f(P_1)+1)$ appears in an arbitrarily taken row of the table.

The first two rows of the table are presented as in Figure 1. Obviously, if $z = 1$, then the number $F_2 = P_1P_2 = P_1\left(\left[\frac{N}{P_1}\right]+1\right)$ is in the second row of the table. If $z > 1$, and has a "polite" divisor (let it be v), then the "problematic" number F_2 is crossed out when we cross out numbers that are multiples of the "polite" v, and the "critical" numbers P_1 and P_2 are now becoming "polite".

Option 2. Consider the "problem" number $F_3 = zP_1P_2$. In the continuation, for the "critical" numbers P_1 and P_2, assume that $z > 1$ and has no "polite" prime divisor. That is, all prime divisors of z are "critical". Given that P_1 is the smallest "critical" prime, then $P_2 > P_1$. There cannot be $z = P_3P_4$, where P_3 and P_4 are both "critical" primes, and P_3 is "increased" in "favor" of P_4 (or vice versa). Figure 1 shows that the number $P_1P_2 = P_1\left(\left[\frac{N}{P_1}\right]+1\right)$ and the number $P_3P_4 = P_3\left(\left[\frac{N}{P_3}\right]+1\right)$ are both in the second row of the table. The conditions $P_1P_2 \geq N+1$, and $P_3P_4 \geq N+1$ are met. It turns out $F_3 \geq (N+1)^2$. And this means that the number $F_3 = P_1P_2P_3P_4$ is outside the table.

Option 3. Next, consider the options if the "increase" of the smallest "critical" number P_1 continues like this: $P_1 \to P_2 \to \cdots \to P_j \to \cdots \to P_n$. And the "problem"

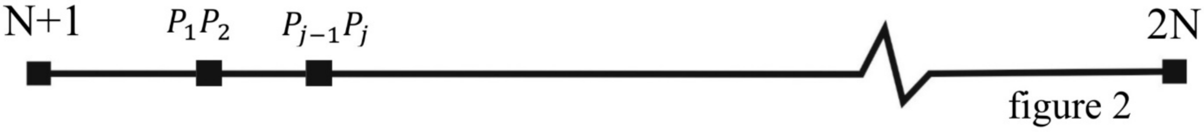

figure 2

number will be like this: $F_4 = P_1 P_2 \cdot ... \cdot P_j \cdot ... \cdot P_n$. There are no "polite" numbers among the natural divisors of the "problematic" number F_4, and the length of the table row (that is, the value of N) is limited. Therefore, the "critical" number P_n must also be "increased". There are two possible options: or $P_n \to P_j$, or $P_n \to P_1$. Here $j = 2, ..., n$, and $n \geq 4$.

Figure 2 shows that regardless of whether $P_1 P_2 > P_{j-1} P_j$ or $P_{j-1} P_j > P_1 P_2$, the conditions $P_1 P_2 \geq N + 1$ and $P_{j-1} P_j \geq N + 1$ are met. So $F_4 \geq (N + 1)^2$ (controversy).

Option 4. Consider the option $F_5 = P_1 P_2 P_3$, where either $P_1 \to P_2 \to P_3 \to P_1$, or $P_1 \to P_2 \to P_3 \to P_2$.

Figure 3 shows a contradiction. According to property 1 in the second line, the number $P_1 P_2$ is the smallest number that is a multiple of the number P_1 (since $P_1 \to P_2$), also, the number $P_2 P_3$ is the smallest number in the second row, which is a multiple of the number P_2. However, these two conditions are contradicted by the location of the numbers $P_3 P_1$ (in the case of $P_3 \to P_1$) and $P_3 P_2$ (in the case of $P_3 \to P_2$), respectively.

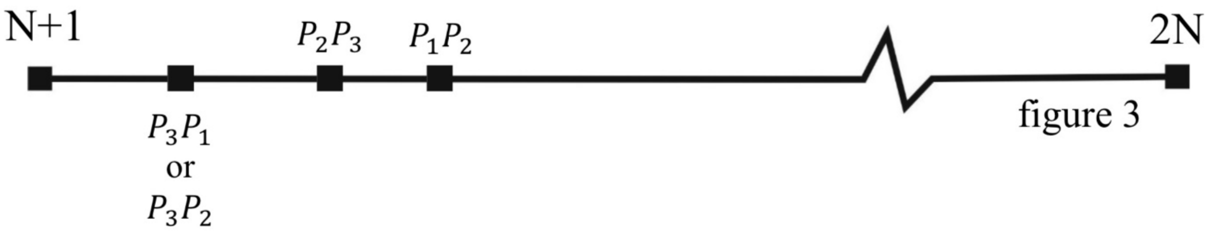

figure 3

Property 2. In the $(g + 1)$ th row of the table, the number g is a "polite" number. In other words, in the first and $(g + 1)$ th rows of the table, the number of numbers that are multiples of g are equal and are in the same columns of the table. It's obvious.

Option 5. Let's consider the case when there is a "problem" number $F_6 = P_1 P_2^2$ in an arbitrarily taken string, where $P_1 \to P_2$ and $P_2 \to P_2$.

Taking into account property 1, it should be $P_1 > P_2$ (figure 4). In other words, in the case of $P_1^2 P_2$, there cannot be a "problem" number in the form of $P_1 > P_2$. Let 's

figure 4

write $P_2P_2 = N + \gamma \underset{\gamma<P_2}{\Longrightarrow} P_1P_2^2 = P_1(N+\gamma) = P_1N + P_1\gamma$. Since the number P_1P_2 in the second row is the smallest multiple of P_1 (property 1), then $P_1\gamma < N$. Therefore, the number $F_6 = P_1P_2^2 = P_1N + P_1\gamma$ is in the row under the number $P_1 + 1$. Given property 2, the number P_1 is "polite", and the number $F_6 = P_1P_2^2$ is not "problematic"

Option 6. Consider the case when there is a "problem" number $F_7 = P_1P_2^2$ in an arbitrarily taken string, where $P_1 \to P_2$ and $P_2 \to P_1$.

figure 5

Let's write (figure 5) $P_1P_2 = N + \gamma \underset{\gamma<P_1 \text{ и } \gamma<P_2}{\Longrightarrow} P_1P_2^2 = P_2(N+\gamma) = P_2N + P_2\gamma$. Since the number P_1P_2 in the second row is the smallest multiple of P_1 and P_2 (property 1) at the same time, then $P_2\gamma < N$. Therefore, the number $F_7 = P_1N + P_1\gamma$ is in the row under the number $P_1 + 1$. Given property 2, the number P_1 is "polite", and the number $F_7 = P_1P_2^2$ is not "problematic".

Option 7. Let's consider the case when there is a "problem" number $F_8 = P_1^{\alpha}$ in an arbitrarily taken string, where $P_1 \to P_1$. It is obvious that $\alpha = 3$.

Otherwise (figure 6), either $F_8 = P_1^4 \geq (N+1)^2$ (the number is outside the table), or the number $F_8 = P_1^2$ is in the second row. So $F_8 = P_1^3$. Let 's write $P_1P_1 = N + \gamma \underset{\gamma<P_1}{\Longrightarrow} P_1^3 = P_1(N+\gamma) = P_1N + P_1\gamma$. Since the number P_1P_1 is the smallest number in the second row, a multiple of P_1 (property 1), then $P_1\gamma < N$. Therefore, the number $F_8 = P_1^3 = P_1N + P_1\gamma$ is in the row numbered $P_1 + 1$. Given property 2, the number P_1 is "polite", and the number $F_8 = P_1^3$ is not "problematic".

The theorem on the distribution of primes is proved.

Corollary 1. Legendre's conjecture. For any natural N between N^2 and $(N+1)^2$ there is at least one prime number**.**

It is obvious that Legendre's hypothesis is a special case of the prime number

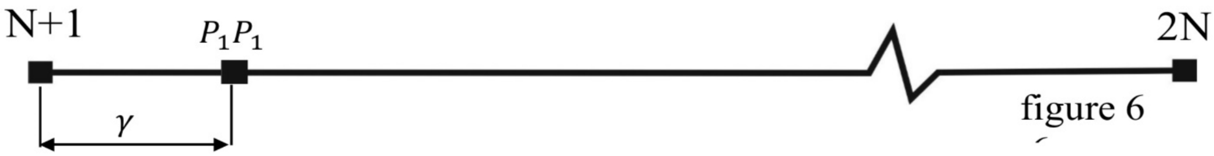

figure 6

distribution theorem, and for any natural N between N^2 and $(N+1)^2$ there will be at least two primes, since there are two complete rows in the specified interval (at least one prime number in each).

Corollary 2. Brocard's conjecture. For any natural number n between p_n^2 and p_{n+1}^2 (where $p_n > 2$ and p_{n+1} are two consecutive primes), there are at least four primes.

For any prime number $p_n > 2$, we can write as follows:

$p_n = N - 1$ and $p_n + 2 = N + 1$.

$p_{n+1} - p_n \geq 2$

Between $p_n^2 = (N-1)^2$ and $(p_n + 2)^2 = (N+1)^2$ there are four complete lines, each of which has at least one prime number. We take into account that the minimum difference between consecutive (starting from 3) primes is 2, and therefore we chose $p_{n+1} = N + 1$. So, the greater the difference between consecutive primes, the more primes there are between their squares.

	$\ldots$	$(N-2)N$
$(N-1)^2$	$\ldots$	$(N-1)N$
$(N-1)N+1$	$\ldots$	N^2
N^2+1	$\ldots$	$(N+1)N$
$(N+1)N+1$	$\ldots$	$(N+2)N$
$(N+1)^2$	$\ldots$	