New theorem on the distribution of prime ideals

Solving the 3rd Landau problem (*Legendre's conjecture*)

Confirmation of the Brocard's conjecture

Confirmation of the Opperman's conjecture

Confirmation of Andrica's conjecture

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Abstract

I am proving a theorem that in each row ($N > 1$ consecutive numbers in each) of a table consisting of a set of natural numbers $[1, 2, 3, \dots, N(N + 2)]$, there is at least one prime number. For the second row of the table, I use Bertrand's postulate, according to which for any natural number $N \geq 2$ there exists a prime number in the range $[N, 2N]$. Next, starting from the 3rd row of the table, I cross out the numbers that are multiples of all the prime numbers (the set L) of the first row: in parallel (simultaneously) in the arbitrary and first rows of the table. In Lemma 2, I prove that there is no chaos in the table as a result of such synchronous deletion of numbers. After proving Lemma 3, it becomes obvious that the presence of the number 1 (one) in the first row indicates that at least one uncrossed number remains in each row of the table – this is a prime number. Using this theorem, we will solve open mathematical problems: Solving the 3rd Landau problem (*Legendre's conjecture*); Confirmation of the Brocard's conjecture; Confirmation of the Opperman's conjecture; Confirmation of Andrica's conjecture.

Keywords

Number theory, prime number, Distribution of prime ideals, Landau's 3rd problem, Brocard's conjecture, Oppermann's conjecture, Andrica's conjecture

Mathematical Subject Classification

11A41, 11R44

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I am an independent mathematician who works outside of academic institutions.

Problem Statement

Let us write the set of natural numbers in the form of a table, where each row contains consecutive (N pieces) numbers (**note: this article does not concern the "Sieve of**

Eratosthenes”). We will prove that each row of this table contains at least one prime number. We will prove the theorem that each row of this table contains at least one prime number. Using this theorem, at the end of the article we will solve several open problems in number theory.

table 1

1	2	3,…	$N-1$	N
$N+1$	$N+2$	$N+3, \dots$	$2N-1$	$2N$
$2N+1$	$2N+2$	$2N+3, \dots$	$3N-1$	$3N$
$3N+1$	$3N+2$	$3N+3, \dots$	$4N-1$	$4N$
…	…	…	…	…
$mN+1$	$mN+2$	$mN+3, \dots$	$(m+1)N-1$	$(m+1)N$
…	…	…	…	…
$(N-1)N+1$	$(N-1)N+2$	$(N-1)N+3, \dots$	N^2-1	N^2
N^2+1	N^2+2	$N^2+3, \dots$	$(N+1)N-1$	$(N+1)N$
$(N+1)N+1$	$(N+1)N+2$	$(N+1)N+3, \dots$	$(N+2)N-1$	$(N+2)N$
$(N+2)N+1=(N+1)^2$				

THEOREM

For any natural numbers $N \geq 2$ and k, where $1 \leq k \leq N+2$, there exists at least one prime number in the interval $[(k-1)N+1, kN]$.

In other words, every full row in the above-described table contains at least one prime number.

PROOF OF THE THEOREM

It is evident that the first row of the table always contains at least one prime number.

According to Bertrand’s Postulate [1], for any natural number $N \geq 2$, there exists a prime number in the interval $[N, 2N]$. Therefore, the second row of the table (for $N \geq 2$) also contains at least one prime number.

Now we will prove that starting from the third row and onward, every arbitrarily selected row in the table contains at least one prime number.

NOTATION

Let's use $t(m) = \left[\frac{N}{m}\right]$ and $T(m)$ to denote the number of multiples of m in the first row of the table before and after the start of the strikeout process, respectively.

Similarly, using $f(m)$ and $F(m)$, we denote the number of multiples of m in a randomly selected row of the table before and after the start of strikeout, respectively (here $m \leq N$).

Then:

$$f(m) = t(m) + \Delta_m \Rightarrow f(m) \geq t(m) \qquad \textbf{(1)}$$

Lemma 1

We prove that either $\Delta_m = 0$, or $\Delta_m = 1$.

PROOF OF LEMMA 1

Let us prove that $\Delta_m < 2$.

Let the length of the first row (i.e., number of elements) be:

$$N = (m-1) + \left(1 + \left(\left[\frac{N}{m}\right] - 1\right) \cdot m\right) + \alpha = \left[\frac{N}{m}\right] \cdot m + \alpha \qquad \textbf{(2)}$$

Where:

$$0 \leq \alpha \leq m-1 \qquad \textbf{(3)}$$

Where:

$(m-1)$ is the number of elements not divisible by m at the start of the first row;

α is the number of all numbers (at the end of the first line) after the largest number, which is a multiple of m ($\alpha = 0$ in the case of $N \equiv 0 (\text{mod } m)$).

Assume the contrary: that there exists a row where $\Delta_m \geq 2$. Then its minimum length would be:

$$N = \left(\left(\left[\frac{N}{m}\right] + \Delta_m\right) - 1\right) \cdot m + 1 = \left(\left(\left[\frac{N}{m}\right] + 2\right) - 1\right) \cdot m + 1 = \left[\frac{N}{m}\right] \cdot m + m + 1 \qquad \textbf{(4)}$$

But using equations (3) and (4), we obtain a contradiction:

$$\left[\frac{N}{m}\right] \cdot m + \alpha = \left[\frac{N}{m}\right] \cdot m + m + 1 \Rightarrow \alpha = m + 1$$

It cannot be $\Delta_m < 0$, since the minimum values of $f(m)$ and $t(m)$ are $\left[\frac{N}{m}\right]$.

LEMMA 1 is proven.

DEFINITIONS

- A number is called a *good* number if $\Delta_m = 0$.
- A number is called a *critical* number if $\Delta_m = 1$.
- If $f(m) = t(m) + 1 = \left[\frac{N}{m}\right] + 1$, we say: the value of m is "*increased in favor of the number*" $\left[\frac{N}{m}\right] + 1$ (let's write it this way $m \to \left[\frac{N}{m}\right] + 1$).
- Similarly, if $F(m) = T(m) + 1$, we say: the value of m is "*increased in favor of the number*" $T(m) + 1$ (let's write it this way $m \to T(m) + 1$).
- If, in an arbitrarily selected row, $\Delta_m = 1$, then that row contains a number F (see (5), called a *problematic* number) divisible by $m\left(\left[\frac{N}{m}\right] + 1\right) = m\left[\frac{N}{m}\right] + m > N$.

$$\begin{cases} F = zm\left(\left[\frac{N}{m}\right]+1\right) = \ zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right]+1\right) \geq z(N+1) \\ m = P_1, \quad \left[\frac{N}{P_1}\right]+1 = P_2, \quad P_1 \to P_2 \Rightarrow P_1 \to \left[\frac{N}{P_1}\right]+1 \Rightarrow P_1P_2 \geq N+1 \end{cases} \qquad \textbf{(5)}$$

P_1, P_2, z – are natural numbers

PROPERTY 1

It is obvious that in the rows with indices $\{1, m+1, 2m+1, \dots\}$, the value $\Delta_m = 0$ remains constant.

Corollary of **PROPERTY 1**

In all the specified rows of the table, m is a good number.

LEMMA 2

Let's assume that we have crossed out in the arbitrarily taken and in the first lines all the numbers that are multiples of the *good* prime $p_1 \in L$, for which the following was true:

$f(p_1) = t(p_1)$.

Let us now analyze the number of remaining (i.e., not eliminated) numbers divisible by some other prime $p_i \in L \setminus p_1$ for which initially:

$f(p_i) = \left[\frac{N}{p_i}\right] + \Delta_{p_i} = t(p_i) + \Delta_{p_i}$

And after crossing out the numbers that are multiples of $p_1 \in L$, we denote the difference $F(p_i) - T(p_i)$ as δ_{p_i}:

$F(p_i) - T(p_i) = \delta_{p_i}$ (see (8))

At the same time, it is obvious that in the first and randomly selected rows there will be no numbers that are multiples of $p_1 p_i$.

Let's prove that:

$\delta_{p_i} \leq \Delta_{p_i}$

PROOF OF LEMMA 2

According to (1), for a prime number $m = p_i$ and for a composite number $m = p_1 p_i$ we write:

$$f(p_i) = t(p_i) + \Delta_{p_i} \qquad \textbf{(6)}$$

$$f(p_1 p_i) \geq t(p_1 p_i) \qquad \textbf{(7)}$$

$$F(p_i) - T(p_i) = \delta_{p_i} \qquad \textbf{(8)}$$

Subtract (7) from (6):

$$f(p_i) - f(p_1 p_i) \leq t(p_i) - t(p_1 p_i) + \Delta_{p_i} \qquad \textbf{(9)}$$

But from definitions:

$f(p_i) - f(p_1 p_i) = F(p_i)$

$t(p_i) - t(p_1 p_i) = T(p_i)$

We substitute the last two equalities in (9) and get

$$F(p_i) \leq T(p_i) + \Delta_{p_i} \quad \textbf{(10)}$$

Compare (8) and (10), we conclude:

$$\delta_{p_i} \leq \Delta_{p_i} \quad \textbf{(11)}$$

LEMMA 2 is proven.

Corollary 1 of **LEMMA 2**

Good numbers do not become *critical* during the process of crossing out.

Corollary 2 of **LEMMA 2**

Suppose that in an arbitrary row $\Delta_m = 1$ (that is $m \rightarrow \left[\frac{N}{m}\right] + 1$). Moreover, if $\left[\frac{N}{m}\right] + 1$ (or one of its multipliers) is a *good* number, then after crossing out the numbers that are multiples of the *good* $\left[\frac{N}{m}\right] + 1$ (or its good divisor), the number m also becomes *good*.

For example, for $N = 13$ in the third row of the table (*table 2*) $\Delta_3 = 1$. In other words, in the first row of such a table, four numbers $3, 6, 9, 12$ are multiples of 3, and in the third row there are five such numbers $27, 30, 33, 36, 39$. That is, $3 \rightarrow \left[\frac{13}{3}\right] + 1 = 5$. The number 5 in this line is a *good* number, that is, $\Delta_5 = 0$. In the third line, we cross out two numbers ($30, 35$) that are multiples of the *good* number 5. In parallel, and in the first line, we cross out two numbers ($5, 10$) that are multiples of the *good* number 5. In the new state of the third row of *table 2*, the number of numbers ($27, 33, 36, 39$) that are multiples of the number 3 has become the same as in the first row ($3, 6, 9, 12$) – four. That is, in the beginning there was $f(3) = \left[\frac{13}{3}\right] + 1 = t(3) + 1 = 4 + 1 = 5$. And after crossing out the numbers that are multiples of 5, for the number 3 it turned out $\delta_3 = 0 \Rightarrow F(3) = T(3) + \delta_3 = T(3) + 0 = 4$.

table 2

1	2	3	4	~~5~~	6	7	8	9	~~10~~	11	12	13
14	15	16	17	18	19	20	21	22	23	24	25	26
27	28	29	~~30~~	31	32	33	34	~~35~~	36	37	38	39

Corollary 3 of **LEMMA 2**

At any stage of deletion, if $\Delta_p = 0$ (or $\delta_p = 0$), then in an arbitrary row of *table 1* we will delete no more numbers of multiples of a *good* p (if any) than in the first row of the table of multiples of p. In this case, there will not be a single multiple of p left in an arbitrary line.

LEMMA 3

If $\Delta_p = 1$, then a critical prime p exists within an arbitrary string, and a *problematic* number F may also be present (see (5)):

$$F = zm\left(\left[\frac{N}{m}\right] + 1\right) = zP_1P_2 = zP_1\left(\left[\frac{N}{P_1}\right] + 1\right) \geq z(N+1)$$

We aim to prove that, after removing all numbers (based on previous results) divisible by the primes in the set $L = \{2,3,5, \dots, P\}$ no *problematic* number F remains in the table.

PROOF OF LEMMA 3

Proof of the contrary. Suppose that after the elimination process, some problematic F numbers remain not crossed out in a randomly selected row. In other words, the number F does not have a good multiplier (divisor). Let's make a table of all possible such problematic numbers. Here $\{P_1, P_2, P_3, P_4\}$ are the set of all possible critical divisors (multipliers) of the number F (*table 3*):.

table 3

Auxiliary lemma 3.1	**Auxiliary lemma 3.2**	**Auxiliary lemma 3.3**	**Auxiliary lemma 3.4**
$F_1 = P_1P_2P_3P_4$	$F_2 = P_1^3$	$F_3 = P_1P_2^2$	$F_4 = P_1P_2P_3$

AUXILIARY LEMMA 3.1

If the problematic number is of the form $F_1 = P_1P_2P_3P_4$, according to (5) for $\{P_1, P_2, P_3, P_4\}$ two variants are valid.

1st option:

$$\begin{cases} P_1 \to P_2, \quad P_2 \to P_1 \\ P_3 \to P_4, \quad P_4 \to P_3 \end{cases} \Rightarrow \begin{cases} P_1 \cdot P_2 \geq N+1 \\ P_3 \cdot P_4 \geq N+1 \end{cases}$$

Therefore,

$$F_1 = P_1P_2P_3P_4 \geq (N+1)^2 \qquad \textbf{(12)}$$

(12) contradicts the assumption, since the number $(N+1)^2$ is outside the table.

2nd option:

$$P_1 \to P_2 \to P_3 \to P_4 \to P_\mu$$

Here $\mu = \{1, 2, 3, 4\}$.

$$P_1 \cdot P_2 \geq N+1, \quad P_3 \cdot P_4 \geq N+1$$

Therefore,

$$F_1 = P_1P_2P_3P_4 \geq (N+1)^2 \qquad \textbf{(13)}$$

(13) contradicts the assumption, since the number $(N+1)^2$ is outside the table.

AUXILIARY LEMMA 3.1 is proven.

AUXILIARY LEMMA 3.2

If the problematic number is of the form $F_2 = P_1^3$, then one option is possible:

$P_1 \to P_1$

Therefore: $P_1 \to P_1 \Rightarrow P_1 = \left[\frac{N}{P_1}\right] + 1 \Rightarrow P_1 \cdot \left(\left[\frac{N}{P_1}\right] + 1\right) = P_1^2 \Rightarrow N + 1 \leq P_1^2 < 2N$

In the second row of the table, the number P_1^2 is the smallest multiple of P_1. Let's write $P_1^2 - P_1 < N$, and continue as follows:

$P_1^2 - N = \gamma < P_1 \Rightarrow \gamma \leq P_1 - 1 \Rightarrow P_1\gamma \leq P_1^2 - P_1 < N \Rightarrow P_1\gamma < N$

Therefore,

$$P_1^2 - N = \gamma \Rightarrow P_1^3 - P_1 N = P_1\gamma \Rightarrow P_1^3 = P_1 N + P_1\gamma \qquad \textbf{(14)}$$

(14) means (since $P_1\gamma < N$) that the number $F_2 = P_1^3 = P_1 N + P_1\gamma$ is in the $(P_1 + 1)$th row of the *table 1*. According to property 1, the number P_1 is *good*, which means that the number $F_2 = P_1^3$ is not *problematic*.

AUXILIARY LEMMA 3.2 is proven.

AUXILIARY LEMMA 3.3

If the problematic number has the form $F_3 = P_1 P_2^2$, then there are four possible options (*table 4)*:

table 4

OPTION A	**OPTION B**	**OPTION C**	**OPTION D**
$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_1$	$P_1 \to P_2 \to P_2$	$P_1 \to P_2 \to P_2$
$P_1 > P_2$	$P_1 < P_2$	$P_1 > P_2$	$P_1 < P_2$

OPTION A. Here, the number $P_1 P_2$ is the smallest number in the second row of the table, a multiple of both P_1 and P_2. So $P_2^2 < N$. Next, $\gamma < P_2 < P_1$ we write $P_1 P_2 = N + \gamma$. Multiply the latter by P_2 and we get:

$F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$

Since $\gamma < P_2$ then $P_2\gamma < N$. This means that the number $F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$ is in the row under the number $(P_2 + 1)$. According to property 1, the number P_2 is *good*, which means that the number $F_3 = P_1 P_2^2$ is not *problematic*.

OPTION B. Here, the number $P_1 P_2$ is the smallest number in the second row of the table, a multiple of both P_1 and P_2. Next, $\gamma < P_1 < P_2$ we write $P_1 P_2 = N + \gamma$. Multiply the latter by P_2 and we get:

$F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$

Since $\gamma < P_1$ then $P_2\gamma < N$. This means that the number $F_3 = P_1 P_2^2 = P_2 N + P_2\gamma$ is in the row under the number $(P_2 + 1)$. According to property 1, the number P_2 is *good*, which means that the number $F_3 = P_1P_2^2$ is not *problematic*.

OPTION C. Here, the number P_2^2 is the smallest number in the second row of the *table 1*, a multiple of P_2. Let's write $P_2^2 = N + \gamma$. Multiply the latter by P_1 and we get:

$P_1P_2^2 = P_1N + P_1\gamma$

Taking into account $\gamma < P_2 < P_1$ we obtain $P_1\gamma < N$. This means that the number $F_3 = P_1P_2^2 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_3 = P_1P_2^2$ is not *problematic*.

OPTION D. Here it turns out that the numbers P_1P_2 and P_2^2 are simultaneously the smallest numbers in the second row that are multiples of P_2. And this is not possible because of $P_1 \neq P_2$.

AUXILIARY LEMMA 3.3 is proven.

AUXILIARY LEMMA 3.4

If the problematic number has the form $F_4 = P_1P_2P_3$, then there are three possible options (*table 5)*:

table 5

OPTION D	**OPTION F**	**OPTION G**
$P_1 \to P_2 \to P_3 \to P_1$	$P_1 \to P_2 \to P_3 \to P_2$	$P_1 \to P_2 \to P_3 \to P_3$

OPTION E. If $P_1 \to P_2 \to P_3 \to P_1$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

** $P_3 \to P_1$. The number P_3P_1 in the second row is the smallest multiple of P_3. It turns out $P_1P_2 < P_3P_1$.

*** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2. It turns out $P_2P_3 < P_1P_2$.

The result is a contradiction:

$$\begin{cases} P_2P_3 < P_1P_2 \Rightarrow P_3 < P_1 \\ P_2P_3 > P_3P_1 \Rightarrow P_2 > P_1 \\ P_1P_2 < P_3P_1 \Rightarrow P_2 < P_3 \end{cases} \begin{cases} P_3 < P_1 < P_2 \Rightarrow P_3 < P_2 \\ \qquad\qquad\qquad\quad P_2 < P_3 \end{cases}$$

$F_4 = P_1P_2P_3$ is not *problematic*.

OPTION F. If $P_1 \to P_2 \to P_3 \to P_2$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

Let's write $P_2P_3 = N + \gamma$.

** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2 (it turns out that $\gamma < P_2$). Hence, $P_2P_3 < P_1P_2$.

*** $P_3 \to P_2$. The number P_2P_3 in the second row is the smallest multiple of P_3 (it turns out that $\gamma < P_3$)..

($P_2P_3 = N + \gamma$) multiply by P_1 and get $P_1P_2P_3 = P_1N + P_1\gamma$. Since $\gamma < P_2,\ \gamma < P_3$, then $P_1\gamma < P_1P_2 \Rightarrow P_1\gamma < N$.

This means that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_4 = P_1P_2P_3$ is not *problematic*.

OPTION G. If $P_1 \to P_2 \to P_3 \to P_3$, then theoretically it turns out:

* $P_1 \to P_2$. The number P_1P_2 in the second row is the smallest multiple of P_1.

** $P_2 \to P_3$. The number P_2P_3 in the second row is the smallest multiple of P_2. Hence, $P_2P_3 < P_1P_2$. We will write $P_2P_3 = N + \gamma$ and multiply by P_1. $\gamma < P_2 \Rightarrow P_1\gamma < N$.

This means that the number $F_4 = P_1P_2P_3 = P_1N + P_1\gamma$ is in the row under the number $(P_1 + 1)$. According to property 1, the number P_1 is *good*, which means that the number $F_4 = P_1P_2P_3$ is not *problematic*.

AUXILIARY LEMMA 3.3 is proven.

LEMMA 3 is proven.

REMARK. In the process of proving *Lemma 3*, it becomes clear that there are no more than two *critical* divisors (multipliers) among the prime divisors of the number F. Therefore, there is a *good* number (*good* numbers) in each row of *table 1*.

THEOREM is proven.

COROLLARY 1. SOLUTION OF THE 3RD LANDAU PROBLEM [2] (Legendre's conjecture). For any natural N between N^2 and $(N+1)^2$ there is at least one prime number**.**

It is obvious that Legendre's hypothesis is a special case of the prime number distribution theorem, and for any natural N between N^2 and $(N+1)^2$ there will be at least two primes, since there are two complete rows in the specified interval (at least one prime number in each) – *table 1*.

COROLLARY 2. BROCARD'S CONJECTURE [3]**.** For any natural number n between p_n^2 and p_{n+1}^2 (where $p_n > 2$ and p_{n+1} are two consecutive primes), there are at least four primes.

For any prime number $p_n > 2$, we can write as follows (*table 6*):

$p_n = N - 1$ and $p_n + 2 = N + 1$.

$p_{n+1} - p_n \geq 2$

Between $p_n^2 = (N-1)^2$ and $(p_n + 2)^2 = (N+1)^2$ there are four complete lines (*table 6*), each of which has at least one prime number. We take into account that the minimum difference between consecutive (starting from 3) primes is 2, and therefore we chose $p_{n+1} = N + 1$. So, the greater the difference between consecutive primes, the more primes there are between their squares.

table 6

	$\dots$	$(N-2)N$
$p_n^2 = (N-1)^2$	$\dots$	$(N-1)N$
$(N-1)N+1$	$\dots$	N^2
N^2+1	$\dots$	$(N+1)N$
$(N+1)N+1$	$\dots$	$(N+2)N$
${p_{n+1}}^2 = (N+1)^2$	$\dots$	

COROLLARY 3. OPPERMAN'S CONJECTURE [4]**.** The conjecture states that, for every integer $N > 1$, there is at least one prime number between $(N-1)N$ and N^2, and at least another prime between N^2 and $N(N+1)$. This is obvious (see *table 6*).

COROLLARY 4. ANDRICA'S CONJECTURE [5]**.** The conjecture states that the inequality $\sqrt{p_{n+1}} - \sqrt{p_n} < 1$ holds for all n, where p_n is the n-th prime number.

There are at least two prime numbers in the range of two full rows of table $[N^2+1; (N+2)N]$. Denote as p_n and p_{n+1}.

If we mark the numbers of two complete rows in the interval $[N^2+1; (N+2)N]$ (*table 6*) on the numeric axis,

$N^2, \dots, p_n, \dots, p_{n+1}, \dots, (N+1)^2-2, (N+1)^2-1$

we will see that the following inequalities are satisfied:

$N^2 < p_n < p_{n+1} < (N+1)^2 - 2 < (N+1)^2 - 1 < (N+1)^2$

This means that the following inequalities also hold:

$\sqrt{N^2} < \sqrt{p_n} < \sqrt{p_{n+1}} < \sqrt{(N+1)^2-2} < \sqrt{(N+1)^2-1} < \sqrt{(N+1)^2}$

We denote: $\sqrt{(N+1)^2} - \sqrt{(N+1)^2-2} = \mu > 0$ **(15)**

Let's write it down:

$\sqrt{N^2} < \sqrt{p_n} < \sqrt{p_{n+1}} < \sqrt{(N+1)^2-2} = \sqrt{(N+1)^2} - \mu$

Obviously, the length of the segment $\left(\sqrt{(N+1)^2} - \mu\right) - \sqrt{N^2}$ is greater than the length of the segment $\sqrt{p_{n+1}} - \sqrt{p_n}$.

$\sqrt{p_{n+1}} - \sqrt{p_n} < \left(\sqrt{(N+1)^2} - \mu\right) - \sqrt{N^2}$

$\sqrt{p_{n+1}} - \sqrt{p_n} < (N+1) - \mu - N$

$\sqrt{p_{n+1}} - \sqrt{p_n} < 1 - \mu$ **(16)**

ANDRICA'S CONJECTURE was confirmed.

Comparing, we can assert that if $p_{n+1} - p_n = 2$, then (15) and (16) are special cases of each other. In other words, the components of both inequalities are a pair of consecutive numbers of the same parity: $[(N+1)^2 - 2, (N+1)^2]$ and $[p_n, p_{n+1}]$.

References

[1] https://en.wikipedia.org/wiki/Bertrand%27s_postulate

[2] https://en.wikipedia.org/wiki/Legendre%27s_conjecture

[3] https://en.wikipedia.org/wiki/Brocard%27s_conjecture

[4] https://en.wikipedia.org/wiki/Oppermann%27s_conjecture

[5] https://en.wikipedia.org/wiki/Andrica%27s_conjecture