$\boldsymbol{abc}$ – CONJECTURE

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ABSTRACT. The subject of this article is the abc _conjecture study. The relevance of the problem under study lies in the fact that it is one of the unsolved problems of number theory [1]. The purpose of the article is to prove that abc-conjecture is correct.

KEY WORDS. *abc* conjecture; Number theory.

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INTRODUCTION

In this paper, two lemmas and one theorem are proved.

LEMMA 1. For any natural number w, there is only a finite number of triples a, b, c of mutually prime natural numbers such as $a + b = c$ and the following inequality holds.

$c > rad(abc)$,

here $rad(abc) = rad(w)$

LEMMA 2. As a consequence of Lemma 1, it becomes obvious that, similarly, for any real positive number ε, there are only a finite number of triples a, b, c of mutually prime natural numbers such as $a + b = c$ and the following inequality holds

$c > rad(abc)^{1+\varepsilon}$

THEOREM. For every positive real number ε, there is a constant $K(\varepsilon)$ such that for all triples a, b, c of mutually prime natural numbers, where $a + b = c$, the following inequality is true

$c < K(\varepsilon) \cdot rad(abc)^{1+\varepsilon}$

PROOFS.

PROOF OF LEMMA 1. We prove that for any natural number w there are only finitely many triples a, b, c of mutually prime natural numbers such as $a + b = c$ and the following inequality

$$c > rad(abc). \quad (1)$$

Here and throughout the article, $rad(abc) = rad(w) = const$ (2)

w is an arbitrary natural number.

Suppose the opposite, that there are an infinite number of triples of a, b, c.

For the numbers a, b, c, we write:

$$\begin{cases} a = \alpha + k_a \cdot rad(abc) \\ b = \beta + k_b \cdot rad(abc) \\ c = \gamma + k_c \cdot rad(abc) \end{cases} \quad (3)$$

Here $\alpha, \beta, \gamma, k_a, k_b, k_c$ are natural numbers, and

$$0 < \alpha < rad(abc), \quad 0 < \beta < rad(abc), \quad 0 < \gamma < rad(abc) \quad (4)$$

NOTE 1. Obviously, if $c = \gamma + k_c \cdot rad(abc)$, then $c > rad(abc)$.

It is obvious from (3) that

$$\begin{cases} rad(abc) \equiv 0\big(mod rad(a)\big) \Rightarrow rad(\alpha) \equiv 0\big(mod rad(a)\big) \Rightarrow rad(\alpha) \geq rad(a) \\ rad(abc) \equiv 0\big(mod rad(b)\big) \Rightarrow rad(\beta) \equiv 0\big(mod rad(b)\big) \Rightarrow rad(\beta) \geq rad(b) \\ rad(abc) \equiv 0\big(mod rad(c)\big) \Rightarrow rad(\gamma) \equiv 0\big(mod rad(c)\big) \Rightarrow rad(\gamma) \geq rad(c) \end{cases} \quad (5)$$

where we take into account that $\alpha \cdot \beta \cdot \gamma \neq 0$, since a, b, c are mutually prime numbers.

From (5) we get

$$rad(\alpha\beta\gamma) \geq rad(abc) \quad \text{and} \quad rad(\alpha\beta\gamma) \equiv 0\big(mod rad(abc)\big) \quad (6)$$

The numbers a, b, c are written as follows (canonical decomposition of the number):

$$\begin{cases} a = a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i} \\ b = b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j} \\ c = c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k} \end{cases} \tag{7}$$

where $a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j, c_1, c_2, \ldots, c_k$ are different prime numbers, $\alpha_1, \alpha_2, \ldots, \alpha_i, \beta_1, \beta_2, \ldots, \beta_j, \gamma_1, \gamma_2, \ldots, \gamma_k$ – natural numbers.

Therefore

$$rad(a) = a_1 \cdot \ldots \cdot a_i \qquad rad(b) = b_1 \cdot \ldots \cdot b_j \qquad rad(c) = c_1 \cdot \ldots \cdot c_k$$

Next, for α, β, γ, taking into account (5) and (6), we write

$$\begin{cases} rad(\alpha) = rad(\partial_\alpha) \cdot a_1.a_2 \cdot \ldots \cdot a_i \\ rad(\beta) = rad(\partial_\beta) \cdot b_1.b_2 \cdot \ldots \cdot b_j \\ rad(\gamma) = rad(\partial_\gamma) \cdot c_1.c_2 \cdot \ldots \cdot c_k \end{cases} \tag{8}$$

where $\partial_\alpha, \partial_\beta, \partial_\gamma$ are natural numbers, and $(a, \partial_\alpha) = 1, (b, \partial_\beta) = 1, (c, \partial_\gamma) = 1$.

NOTE 2. If $rad(\alpha) = rad(a)$, then $\partial_\alpha = 1$, if $rad(\beta) = rad(b)$, then $\partial_\beta = 1$ and if $rad(\gamma) = rad(c)$, then $\partial_\gamma = 1$.

NOTE 3. According to the conditions of the problem, the values of the numbers a, b, c change (increase), and $rad(abc) = const$.

NOTE 4. Lots of different triples of numbers (regardless of the growth of the values of a, b, c) α, β, γ is limited, since $\alpha, \beta, \gamma < rad(abc) = const$.

Taking into account $a + b = c$, we write

$$a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i} + b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j} = c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k} \Rightarrow$$

$$\Rightarrow \frac{a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} + \frac{b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} = 1 \tag{9}$$

The terms on the left side of equality (9) are written in the form of formulas of numerical sequences:

$$0 < A(n) = \frac{a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} < 1, \qquad 0 < B(n) = \frac{b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} < 1 \tag{10}$$

Here $n = \{1,2,3,\dots\}$. Suppose that $a \to +\infty, b \to +\infty, c \to +\infty$. In this case, the values of the numbers a, b, c will grow only due to the growth of degrees $\left(\alpha_1, \alpha_2, \dots, \alpha_i, \beta_1, \beta_2, \dots, \beta_j, \gamma_1, \gamma_2, \dots, \gamma_k\right)$, since $rad(abc) = const$. In this case, it may be that the values of some degrees will remain limited (they do not tend to infinity), and for the same reason fractions $\frac{P_a}{P_c}$ and $\frac{P_b}{P_c}$ are formed. That is

$$\begin{cases} a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \dots \cdot a_i^{\alpha_i} = a_{01}^{\alpha_{01}} \cdot a_{02}^{\alpha_{02}} \cdot \dots \cdot a_{0w}^{\alpha_{0w}} \cdot a_{11}^{\alpha_{11}} \cdot a_{12}^{\alpha_{12}} \cdot \dots \cdot a_{1v}^{\alpha_{1v}} \\ b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \dots \cdot b_j^{\beta_j} = b_{01}^{\beta_{01}} \cdot b_{02}^{\beta_{02}} \cdot \dots \cdot b_{0d}^{\beta_{0d}} \cdot b_{11}^{\beta_{11}} \cdot b_{12}^{\beta_{12}} \cdot \dots \cdot b_{1t}^{\beta_{1t}} \\ c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \dots \cdot c_k^{\gamma_k} = c_{01}^{\gamma_{01}} \cdot c_{02}^{\gamma_{02}} \cdot \dots \cdot c_{0m}^{\gamma_{0m}} \cdot c_{11}^{\gamma_{11}} \cdot c_{12}^{\gamma_{12}} \cdot \dots \cdot c_{1n}^{\gamma_{1n}} \end{cases} \tag{11}$$

Here

$$\begin{cases} w + v = i \\ d + t = j \\ m + n = k \end{cases} \tag{12}$$

$$\begin{cases} \left\{a_1^{\alpha_1}, a_2^{\alpha_2}, \dots, a_i^{\alpha_i}\right\} = \left\{a_{01}^{\alpha_{01}}, a_{02}^{\alpha_{02}}, \dots, a_{0w}^{\alpha_{0w}}\right\} \cup \left\{a_{11}^{\alpha_{11}} \cdot a_{12}^{\alpha_{12}} \cdot \dots \cdot a_{1v}^{\alpha_{1v}}\right\} \\ \left\{b_1^{\beta_1}, b_2^{\beta_2}, \dots, b_j^{\beta_j}\right\} = \left\{b_{01}^{\beta_{01}}, b_{02}^{\beta_{02}}, \dots, b_{0d}^{\beta_{0d}}\right\} \cup \left\{b_{11}^{\beta_{11}}, b_{12}^{\beta_{12}}, \dots, b_{1t}^{\beta_{1t}}\right\} \\ \left\{c_1^{\gamma_1}, c_2^{\gamma_2}, \dots, c_k^{\gamma_k}\right\} = \left\{c_{01}^{\gamma_{01}}, c_{02}^{\gamma_{02}}, \dots, c_{0m}^{\gamma_{0m}}\right\} \cup \left\{c_{11}^{\gamma_{11}}, c_{12}^{\gamma_{12}}, \dots, c_{1n}^{\gamma_{1n}}\right\} \end{cases} \tag{13}$$

$$\begin{cases} a_{01}^{\alpha_{01}} \cdot a_{02}^{\alpha_{02}} \cdot \dots \cdot a_{0w}^{\alpha_{0w}} = P_a \\ b_{01}^{\beta_{01}} \cdot b_{02}^{\beta_{02}} \cdot \dots \cdot b_{0d}^{\beta_{0d}} = P_b \\ c_{01}^{\gamma_{01}} \cdot c_{02}^{\gamma_{02}} \cdot \dots \cdot c_{0m}^{\gamma_{0m}} = P_c \end{cases} \qquad \begin{cases} a_{11}^{\alpha_{11}} \cdot a_{12}^{\alpha_{12}} \cdot \dots \cdot a_{1v}^{\alpha_{1v}} = M_a^{f(\alpha)} \\ b_{11}^{\beta_{11}} \cdot b_{12}^{\beta_{12}} \cdot \dots \cdot b_{1t}^{\beta_{1t}} = M_b^{f(\beta)} \\ c_{11}^{\gamma_{11}} \cdot c_{12}^{\gamma_{12}} \cdot \dots \cdot c_{1n}^{\gamma_{1n}} = M_c^{f(\gamma)} \end{cases} \tag{14}$$

$$\begin{cases} a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \dots \cdot a_i^{\alpha_i} = P_a \cdot M_a^{f(\alpha)} \\ b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \dots \cdot b_j^{\beta_j} = P_b \cdot M_b^{f(\beta)} \\ c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \dots \cdot c_k^{\gamma_k} = P_c \cdot M_c^{f(\gamma)} \end{cases} \tag{15}$$

Let's write equality (9) as follows

$$\frac{P_a \cdot M_a^{f(\alpha)}}{P_c \cdot M_c^{f(\gamma)}} + \frac{P_b \cdot M_b^{f(\beta)}}{P_c \cdot M_c^{f(\gamma)}} = 1 \Rightarrow \frac{P_a}{P_c} \cdot \frac{M_a^{f(\alpha)}}{M_c^{f(\gamma)}} + \frac{P_b}{P_c} \cdot \frac{M_b^{f(\beta)}}{M_c^{f(\gamma)}} = 1 \tag{16}$$

If $P_a M_a^{f(\alpha)} \to +\infty$, $P_b \cdot M_b^{f(\beta)} \to +\infty$, $P_c \cdot M_c^{f(\gamma)} \to +\infty$, then taking into account (9) and (16) we come to the conclusion that $f(\alpha), f(\beta), f(\gamma)$ infinitely large functions of the same order. Please note that $P_a,\ M_a,\ P_b,\ M_b,\ P_c,\ M_c$ limited natural numbers.

Let's clarify that:

M_a – the product of those prime divisors of a whose degrees tend to infinity;

M_b – the product of those prime divisors of b whose degrees tend to infinity;

M_c – the product of those prime divisors of c whose degrees tend to infinity.

P_a, P_b, P_c – the explanation of these components is in (14).

For (16), let's analyze four possible options:

1. Since the numbers a, b, c are mutually prime, then $\frac{M_a}{M_c} \neq 1,\ \frac{M_b}{M_c} \neq 1$;
2. If $\frac{M_a}{M_c} > 1$ and/or $\frac{M_b}{M_c} > 1$, then the left sides of equalities (16) and (9) tend to infinity;
3. If $\frac{M_a}{M_c} < 1$ and/or $\frac{M_b}{M_c} < 1$, then the left sides of equalities (16) and (9) tend to zero.
4. It can be assumed that

$$\begin{cases} a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i} = \left(F(a)\right)^{g(c)} \\ b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j} = \left(H(b)\right)^{g(c)} \\ c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k} = \left(g(c)\right)^{g(c)} \end{cases} \tag{17}$$

where $F(a),\ H(b),\ g(c)$, are infinitely large functions of the same order, and $g(c) > F(a),\ g(c) > H(b)$ (18)

In this case, (9) we write as follows

$$\frac{a_1^{\alpha_1} \cdot a_2^{\alpha_2} \cdot \ldots \cdot a_i^{\alpha_i}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} + \frac{b_1^{\beta_1} \cdot b_2^{\beta_2} \cdot \ldots \cdot b_j^{\beta_j}}{c_1^{\gamma_1} \cdot c_2^{\gamma_2} \cdot \ldots \cdot c_k^{\gamma_k}} = 1 \Rightarrow \frac{\left(F(a)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} + \frac{\left(H(b)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} = 1 \tag{19}$$

For the left part (19), we write

$$\frac{\left(F(a)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} \leq \frac{\left(g(c)-1\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} = \left(\frac{g(c)-1}{g(c)}\right)^{g(c)} =$$

$$= \left(1 - \frac{1}{g(c)}\right)^{g(c)} \Rightarrow \lim_{g(c)\to+\infty} \left(1 - \frac{1}{g(c)}\right)^{g(c)} = \frac{1}{e} < \frac{1}{2} \qquad (20)$$

$$\frac{\left(H(b)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} \le \frac{(g(c)-1)^{g(c)}}{\left(g(c)\right)^{g(c)}} = \left(\frac{g(c)-1}{g(c)}\right)^{g(c)} =$$

$$= \left(1 - \frac{1}{g(c)}\right)^{g(c)} \Rightarrow \lim_{g(c)\to+\infty} \left(1 - \frac{1}{g(c)}\right)^{g(c)} = \frac{1}{e} < \frac{1}{2} \qquad (21)$$

(20) and (21) contradict (19) and (9), that is

$$\frac{\left(F(a)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} + \frac{\left(H(b)\right)^{g(c)}}{\left(g(c)\right)^{g(c)}} < \frac{1}{e} + \frac{1}{e} < 1 \qquad (22)$$

The analysis of all four variants of equality (16) confirms that for any natural number w, the value of the number c (and the number of such numbers) given by conditions (1) and (2) are limited. So there is a number C such that

$$C > c > rad(abc) \qquad (23)$$

Lemma 1 has been proved.

PROOF OF LEMMA 2. We prove that for every real positive number ε there are only a finite number of triples a, b, c of mutually prime natural numbers such as $a + b = c$ and the following inequality holds

$$c > rad(abc)^{1+\varepsilon} \qquad (24)$$

It is obvious that the appearance in (1) of any real positive number ε (as indicated in (24)) strengthens lemma 1, reduces the value of the number c, and the inequality is valid

$$C > c > rad(abc)^{1+\varepsilon} \qquad (25)$$

Lemma 2 has been proved.

PROOF OF THE THEOREM. Если $C > c > rad(abc)^{1+\varepsilon}$, то это означает, что существует число μ, для которого выполняется условие

$$c = \mu rad(abc)^{1+\varepsilon} \qquad (26)$$

It becomes obvious that for every real positive number ε there is a constant $K(\varepsilon) > \mu$, such that for any triple of mutually prime positive integers a, b, c such as $a + b = c$, the following inequality is true

$$c < K(\varepsilon) \cdot rad(abc)^{1+\varepsilon} \quad (27)$$

The theorem is proved.

$\boldsymbol{abc}$ **– CONJECTURE** is correct**.**

LIST OF LITERATURE

1. https://en.wikipedia.org/wiki/Abc_conjecture